Chapter 3

The Buckingham Π-theorem and the dimension matrix

Learning outcomes

By the end of this chapter you should be able to:

- state and apply the Buckingham Π-theorem to determine the number of dimensionless groups;
- construct and use the dimension matrix to find the dimensionless complexes systematically;
- interpret the physical meaning of dimensionless groups such as the Reynolds number and the drag coefficient.

3.1 Motivating example: the undamped pendulum

In the previous chapter you met an example of a damped pendulum, we begin this chapter by exploring a simpler problem, the undamped pendulum, in more detail. Consider a simple pendulum: a weight of mass m suspended from a rigid, massless rod of length l, free to swing in a vertical plane under the influence of gravity. We wish to determine the period of oscillation τ.

Let θ denote the angle between the rod and the downward vertical. Newton's second law applied to the tangential component of the bob's motion gives the equation of motion

$$ml\ddot{\theta} = -mg\sin\theta,$$

where g is the gravitational acceleration and $\ddot{\theta} = \mathrm{d}^2\theta/\mathrm{d}t^2$. The mass m cancels immediately, yielding

$$\ddot{\theta} = -\frac{g}{l}\sin\theta. \tag{3.1}$$

This already tells us something important: the motion is independent of the bob's mass. The governing parameters that remain are l, g and the initial swing angle, α, (the amplitude). For small oscillations, $\sin\theta \approx \theta$, and (3.1) reduces to the equation of simple harmonic motion,

$$\ddot{\theta} = -\frac{g}{l}\,\theta,$$

whose solution is $\theta(t) = \alpha\cos(\omega t)$ with angular frequency $\omega = \sqrt{g/l}$. The period is therefore

$$\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}.$$

We can arrive at essentially the same result (up to the dimensionless numerical prefactor 2π) without solving the differential equation at all, using only dimensional considerations. Since m has cancelled, the period τ can depend only on l and g (ignoring the angle α for the moment). The dimensions of these quantities are

$$[\tau] = \mathrm{T}, \qquad [l] = \mathrm{L}, \qquad [g] = \mathrm{L}\,\mathrm{T}^{-2}.$$

The only combination of l and g that has the dimension of time is $\sqrt{l/g}$:

$$\left[\sqrt{\frac{l}{g}}\right] = \left(\frac{\mathrm{L}}{\mathrm{L}\,\mathrm{T}^{-2}}\right)^{1/2} = \mathrm{T}.$$

Hence τ must be proportional to $\sqrt{l/g}$, and we can write

$$\tau = C\sqrt{\frac{l}{g}},$$

where C is a dimensionless constant that dimensional analysis alone cannot determine. Comparison with the exact small-angle solution shows that $C = 2\pi$.

Key idea

By demanding that a physical relation be dimensionally consistent, we can deduce the functional form of the answer up to a dimensionless prefactor, without solving any equations. Here we could spot the right combination by inspection because there were only two parameters. In the next section we turn this insight into a systematic recipe, the *dimension matrix*, that works even when there are many parameters and the answer is not obvious.

3.2 A systematic method: the dimension matrix

For the pendulum we found the right combination by inspection, because there were only two parameters and a single dimension (time) to match. With more parameters this guesswork quickly becomes hopeless. Fortunately there is a mechanical procedure that always works. We develop it here on the *same* pendulum problem, so you can see the machinery reproduce an answer you already trust, and then we apply it to harder problems in the worked examples that follow.

The procedure has three practical steps; the third is where the dimension matrix does the work.

Step 1: Identify the governing parameters

This first step requires intuition, common sense, background knowledge and a little creativity. The governing parameters may include:

(i) Directly measurable quantities and those to be determined (e.g. the period, length and bob mass of a pendulum).

(ii) Quantities we believe (or have good reason to consider) important. For the pendulum these might be a friction parameter or the swing angle α. Very often, the list of important parameters emerges gradually, by trial and error.

(iii) Relevant fundamental constants. For example, include g or G if gravity is important, $\hbar$ to allow for quantum effects, c (the speed of light) for relativistic effects, and so on.

Pendulum: the governing parameters are τ, l, g and m, so $n = 4$.

Step 2: Count the parameters with independent dimensions

A set of parameters has **independent dimensions** if none of their dimensions can be built from the dimensions of the others; equivalently, they cannot be combined into a dimensionless quantity. For example, l, τ and m have independent dimensions, but l, τ and g do *not*, because

$$\tau\sqrt{\frac{g}{l}}, \qquad \frac{\tau^2 g}{l}, \qquad \left(\frac{\tau^2 g}{l}\right)^p$$

are dimensionless for any power p.

Call the number of parameters with independent dimensions k. It can never exceed the number of basic units in play. In this course the basic units are M (mass), L (length), T (time) and Θ (temperature), so k will not normally exceed four, and is often smaller. *Pendulum:* the temperature Θ does not appear, so $k = 3$ (the units M, L and T). If we anticipate that the period cannot depend on the bob's mass and drop m, then $k = 2$ (only L and T remain).

The number of independent dimensionless groups we can form is then

$$m = n - k.$$

For the pendulum, $m = 4 - 3 = 1$: there is a single dimensionless group to find. (We state this counting rule precisely as the Π-theorem in the next section; for now let us use it.)

Step 3: Build the dimension matrix and solve

Because the dimensions of every parameter are power-law monomials of the basic units, the dimensionless group Π must itself be a power-law monomial of the parameters:

$$\Pi = \tau^{k_1}\, l^{k_2}\, m^{k_3}\, g^{k_4}, \tag{3.2}$$

with the exponents k_1, k_2, k_3, k_4 chosen so that Π is dimensionless,

$$[\Pi] = 1 = \mathrm{M}^0\,\mathrm{L}^0\,\mathrm{T}^0.$$

To find these exponents, write down the **dimension matrix**, whose columns are the parameters and whose rows are the basic units:

	τ (k_1)	l (k_2)	m (k_3)	g (k_4)
M	0	0	1	0
L	0	1	0	1
T	1	0	0	-2

Each entry is the exponent of that basic unit in the dimension of that parameter. Reading the columns recovers the dimensions we started with:

$$[\tau] = \mathrm{T}^1, \qquad [l] = \mathrm{L}^1, \qquad [m] = \mathrm{M}^1, \qquad [g] = \mathrm{L}^1\,\mathrm{T}^{-2}.$$

Now read the *rows*. Demanding that the total exponent of each basic unit in (3.2) vanishes gives one linear equation per row:

$$\begin{aligned} \mathrm{M}: &\quad 0\,k_1 + 0\,k_2 + 1\,k_3 + 0\,k_4 = 0, \\ \mathrm{L}: &\quad 0\,k_1 + 1\,k_2 + 0\,k_3 + 1\,k_4 = 0, \\ \mathrm{T}: &\quad 1\,k_1 + 0\,k_2 + 0\,k_3 - 2\,k_4 = 0. \end{aligned}$$

These are three equations for four unknowns, one fewer equation than unknown, which is exactly the single degree of freedom ($m = n - k = 1$) the counting rule promised. We are therefore free to fix one exponent; since we want a relation of the form $\tau = F(\ldots)$, take

$$k_1 = 1.$$

The first equation gives $k_3 = 0$ (the bob mass does not enter, as expected), and the remaining two reduce to

$$k_2 + k_4 = 0, \qquad k_1 - 2k_4 = 0,$$

so that $k_4 = \frac{1}{2}$ and $k_2 = -\frac{1}{2}$. Hence

$$\Pi = \tau\, l^{-1/2}\, g^{1/2},$$

and setting Π equal to a constant recovers

$$\boxed{\tau = \Pi\sqrt{\frac{l}{g}},}$$

exactly the form we obtained by inspection, now produced by a procedure that will scale to much harder problems.[1]

It is worth pausing on what just happened: the Π-theorem is simply the statement that *any* equation can be rewritten in dimensionless form using units natural to the problem, and it tells us in advance *how many* dimensionless parameters will appear.

3.3 The Π-theorem (Buckingham's theorem)

The counting rule we used above has a name and a precise statement.

Key idea

Buckingham's Π-theorem. Suppose a problem involves n governing parameters, of which k have independent dimensions. Then the relationship between them can be rewritten as a relationship between exactly

$$m = n - k$$

dimensionless parameters $\Pi_1, \Pi_2, \ldots, \Pi_m$:

$$\Phi(\Pi_1, \Pi_2, \ldots, \Pi_m) = 0,$$

for some function Φ. We do not need to know the form of Φ; the point is that such a relation exists, and that it involves far fewer arguments than we started with.[a]

[a] The theorem is due to Edgar Buckingham (1867–1940), American physicist, in 1914.

The reduction from n parameters to $m = n - k$ dimensionless ones, even when $k \le 4$, is an enormous practical saving. To appreciate why, imagine trying to record a relationship as a table of numbers: a function of one variable fits on a single line, a function of two variables fills a page, a function of three variables fills a book, and a function of four variables would fill an entire library.[2] Cutting the number of arguments is therefore the difference between a tractable problem and an impossible one.

[1] For the curious: the number of parameters with independent dimensions, k, always equals the *rank* of the dimension matrix. If you ever find the rank is smaller than your count, you have over-counted the independent parameters and should recount, an easy slip to correct.

[2] S. J. Kline, *Similitude and Approximation Theory*.

Optional — where the theorem comes from

You can skip this box on a first reading; it explains *why* the counting rule holds, and is not needed to apply the dimension matrix.

Write the relationship between all the parameters in the generic form

$$f(a_1, a_2, \ldots, a_k,\, b_1, b_2, \ldots, b_m) = 0,$$

where $a_1, \ldots, a_k$ are the parameters with independent dimensions and $b_1, \ldots, b_m$ are the rest. By definition, the dimensions of each b_i can be expressed through those of the a_j:

$$[b_i] = [a_1]^{p_i}[a_2]^{q_i} \cdots [a_k]^{r_i}, \qquad i = 1, \ldots, m.$$

The key idea is that the relation $f = 0$ cannot depend on our arbitrary choice of units. We are therefore free to choose units that make a_1 equal to one of its own units, that is to *rescale*, so that the numerical value of a_1 becomes 1. (Choosing the metre as your unit of length makes a one-metre rod have numerical length 1; nothing physical changes.) After this rescaling the first argument is simply the pure number 1:

$$f(1, \ldots) = 0.$$

Now rescale again to make a_2 have unit magnitude. The first argument, already a pure number, is unaffected; the others may change:

$$f(1, 1, \ldots) = 0.$$

Repeating this exactly k times turns the first k arguments into the number 1, while each remaining argument is left as a dimensionless combination Π_i of the original parameters. The result is

$$f(1, 1, \ldots, 1, \Pi_1, \ldots, \Pi_m) = 0 \qquad \Longleftrightarrow \qquad \Phi(\Pi_1, \ldots, \Pi_m) = 0,$$

which is the theorem. Notice that this also explains why none of the chosen a_i can itself be dimensionless: a dimensionless parameter has no units to rescale, so it could never be reduced to 1 this way.

Pendulum. With $a_1 = \tau$, $a_2 = m$, $a_3 = l$ and $b_1 = g$, successive rescalings give

$$f(1, m, l, g\tau^2) = 0, \quad f(1, 1, l, g\tau^2) = 0, \quad f\left(1, 1, 1, \tfrac{g\tau^2}{l}\right) = 0,$$

so the single dimensionless group is $\Pi_1 = g\tau^2/l$, the square of the group $\tau\sqrt{g/l}$ we found with the dimension matrix, and equally valid (any power of a dimensionless group is again dimensionless).

3.4 Worked example: terminal velocity of a raindrop

A raindrop of mass m falls through air under gravity until it reaches a constant speed, the **terminal velocity** v. At terminal velocity the gravitational force is exactly balanced by air resistance, and the drop no longer accelerates. We wish to determine how v depends on the governing parameters using the dimension matrix.

The relevant governing parameters are:

- v, the terminal velocity;

- m, the mass of the drop;
- g, the gravitational acceleration;
- ρ, the density of air;
- A, the cross-sectional area of the drop.

Step 1. There are $n = 5$ governing parameters. Their dimensions are:

$$[v] = \mathrm{L}\,\mathrm{T}^{-1}, \qquad [m] = \mathrm{M}, \qquad [g] = \mathrm{L}\,\mathrm{T}^{-2}, \qquad [\rho] = \mathrm{M}\,\mathrm{L}^{-3}, \qquad [A] = \mathrm{L}^2.$$

There are $k = 3$ base dimensions (M, L, T), so the Π-theorem predicts $m = n-k = 5-3 = 2$ dimensionless groups.

However, we can simplify. At terminal velocity the weight mg is balanced by drag, so the combination mg always appears together as a single force. Physically, only the *weight* $W = mg$ matters, not m and g separately. Replacing them with W reduces the parameter list to four: v, W, ρ and A.

Step 2 (revised). Now $n = 4$ governing parameters with dimensions:

$$[v] = \mathrm{L}\,\mathrm{T}^{-1}, \qquad [W] = \mathrm{M}\,\mathrm{L}\,\mathrm{T}^{-2}, \qquad [\rho] = \mathrm{M}\,\mathrm{L}^{-3}, \qquad [A] = \mathrm{L}^2.$$

Still $k = 3$ base dimensions, so $m = n - k = 4 - 3 = 1$, a single dimensionless group. The dimension matrix is:

	v (k_1)	W (k_2)	ρ (k_3)	A (k_4)
M	0	1	1	0
L	1	1	-3	2
T	-1	-2	0	0

Step 3. We seek $\Pi = v^{k_1} W^{k_2} \rho^{k_3} A^{k_4}$ with $[\Pi] = 1$. The equations for k_i are:

$$\begin{aligned} \mathrm{M}: &\quad k_2 + k_3 = 0, \\ \mathrm{L}: &\quad k_1 + k_2 - 3k_3 + 2k_4 = 0, \\ \mathrm{T}: &\quad -k_1 - 2k_2 = 0. \end{aligned}$$

Since we want v as a function of the other parameters, set $k_1 = 1$. Then:

$$\begin{aligned} \mathrm{T}: &\quad k_2 = -\tfrac{1}{2}k_1 = -\tfrac{1}{2}, \\ \mathrm{M}: &\quad k_3 = -k_2 = \tfrac{1}{2}, \\ \mathrm{L}: &\quad 1 - \tfrac{1}{2} - \tfrac{3}{2} + 2k_4 = 0 \quad \Rightarrow \quad k_4 = \tfrac{1}{2}. \end{aligned}$$

Step 4. The single dimensionless group is:

$$\Pi = v\, W^{-1/2}\, \rho^{1/2}\, A^{1/2} = v\sqrt{\frac{\rho A}{W}}.$$

Since $m = 1$, the Π-theorem gives $\Pi = \text{const}$, i.e.

$$\boxed{v = C\sqrt{\frac{W}{\rho A}} = C\sqrt{\frac{mg}{\rho A}},}$$

where C is a dimensionless constant that must be determined experimentally or by solving the full fluid mechanics problem.

Key idea

This result tells us that heavier drops (larger W) fall faster, and denser air (larger ρ) slows them down, both physically intuitive. The cross-sectional area A also appears: a larger drop presents more area to the air and so experiences more drag. Since a spherical drop of radius r has $m \propto r^3$ and $A \propto r^2$, we find $v \propto \sqrt{r}$, larger raindrops fall faster, which is why you feel large drops hit harder than drizzle.

3.5 Worked example: drag force on a sphere

We shall now consider a slightly more complicated example which will give rise to two dimensionless complexes rather than one. The drag force F_D on a sphere depends on its diameter L, velocity V, the fluid density ρ, and the dynamic viscosity μ. This is the foundation of fluid dynamic similarity.

Step 1: There are $n = 5$ governing parameters: F_D, L, V, ρ and μ. Their dimensions are:

$$[F_D] = \mathrm{M\,L\,T^{-2}}, \qquad [L] = \mathrm{L}, \qquad [V] = \mathrm{L\,T^{-1}}, \qquad [\rho] = \mathrm{M\,L^{-3}}, \qquad [\mu] = \mathrm{M\,L^{-1}\,T^{-1}}.$$

There are $k = 3$ base dimensions: M, L and T. Hence there are $m = n - k = 5 - 3 = 2$ dimensionless complexes.

Step 2: The dimension matrix has the form

	F_D (k_1)	L (k_2)	V (k_3)	ρ (k_4)	μ (k_5)
M	1	0	0	1	1
L	1	1	1	−3	−1
T	−2	0	−1	0	−1

and we seek $\Pi = F_D^{k_1} L^{k_2} V^{k_3} \rho^{k_4} \mu^{k_5}$ with $[\Pi] = 1$. The equations for k_i follow as

$$\begin{aligned} \mathrm{M}: &\quad k_1 + k_4 + k_5 = 0, \\ \mathrm{L}: &\quad k_1 + k_2 + k_3 - 3k_4 - k_5 = 0, \\ \mathrm{T}: &\quad -2k_1 - k_3 - k_5 = 0, \end{aligned}$$

three equations for five unknowns. Since $m = 2$, two of the exponents can be chosen freely.

Step 3: first Π-group. Put $k_1 = 1$ and $k_5 = 0$. Then

$$\begin{aligned} \mathrm{M}: &\quad k_4 = -k_1 = -1, \\ \mathrm{T}: &\quad k_3 = -2k_1 = -2, \\ \mathrm{L}: &\quad 1 + k_2 - 2 + 3 = 0 \quad \Rightarrow \quad k_2 = -2, \end{aligned}$$

which gives the first dimensionless complex:

$$\Pi_1 = \frac{F_D}{\rho\, V^2\, L^2}.$$

This is recognised as the **drag coefficient** C_D (up to a conventional numerical factor).

Step 4: second Π-group. Put $k_1 = 0$ and $k_5 = 1$. Then

$$\begin{aligned} \mathrm{M}: &\quad k_4 = -k_5 = -1, \\ \mathrm{T}: &\quad k_3 = -k_5 = -1, \\ \mathrm{L}: &\quad k_2 - 1 + 3 - 1 = 0 \quad \Rightarrow \quad k_2 = -1, \end{aligned}$$

which gives the second dimensionless complex:

$$\Pi_2 = \frac{\mu}{\rho V L}.$$

The reciprocal of Π_2 is the **Reynolds number**, $\mathrm{Re} = \rho V L/\mu$.

Result. The Π-theorem tells us that

$$\boxed{\Phi(\Pi_1, \Pi_2) = 0, \qquad \text{or equivalently} \qquad C_D = C_D(\mathrm{Re}).}$$

Key idea

Instead of the original five-parameter problem, the drag on a sphere is fully characterised by a single curve $C_D(\mathrm{Re})$, which can be determined experimentally or computationally once and for all. This result underpins the principle of **dynamic similarity** in fluid mechanics: two flows past geometrically similar bodies at the same Reynolds number will have the same drag coefficient, regardless of the actual values of L, V, ρ and μ.

Exercise 3.1

A planet moves in a circular orbit of radius R around the Sun. We want to find how its orbital period τ (the time for one complete orbit) depends on R, the mass M of the Sun, and the gravitational constant G.

(a) Newton's law of gravitation says that the force between masses M and m a distance r apart is $F = GMm/r^2$. Use this to show that $[G] = \mathrm{M}^{-1}\,\mathrm{L}^3\,\mathrm{T}^{-2}$.

(b) Write down the dimensions of τ, R and M. How many dimensionless groups does the Π-theorem predict?

(c) Construct the dimension matrix and solve for the exponents to show that

$$\tau = C\sqrt{\frac{R^3}{GM}},$$

where C is a dimensionless constant.

(d) Deduce that $\tau^2 \propto R^3$ for all planets orbiting the Sun. This is Kepler's third law, which Kepler found from observations in 1619, long before Newton.

(e) The Earth orbits at $R = 1$ astronomical unit (AU) with a period of 1 year. Mars orbits at $R = 1.52\,\mathrm{AU}$. Predict the length of a Martian year, without knowing C, G or M.

(f) Why does the mass of the planet itself not appear in the answer? What would change if you included it in the list of parameters?

Exercise 3.2

The period of a water wave depends on its wavelength λ, the water depth h, the gravitational acceleration g, the water density ρ, and the surface tension σ.

(a) Write down the dimensions of each variable.

(b) How many dimensionless groups does the Π-theorem predict?

(c) Construct the dimension matrix and find the dimensionless groups.

Worked example: the Sedov–Taylor blast wave

A nuclear explosion can be considered with an appropriate and reasonable idealisation as an instantaneous release of a certain energy, E, which causes an expanding shock wave to propagate outwards. As the amount of energy released is very large, pressure inside the expanding, very hot region is by far larger than that in the surrounding medium, say, air. This leads to a rapid expansion of the hot gas region, the fireball. Since the expansion is so fast, the gas energy lost during the expansion at this early stage is negligible, so $E = \text{const}$ is a good approximation at early stages of the expansion, where E is the total energy of the fireball. Obviously, this energy is very close to the energy released in the explosion: the original air energy within the heated region is negligible in comparison with the explosion energy. However, the ambient gas mass density ρ cannot be neglected because the mass of the gas produced in the explosion (the explosion debris) is negligible and the hot air is largely the ambient air heated by the radiation produced in the explosion and swept up by the outer boundary of the expanding hot region.

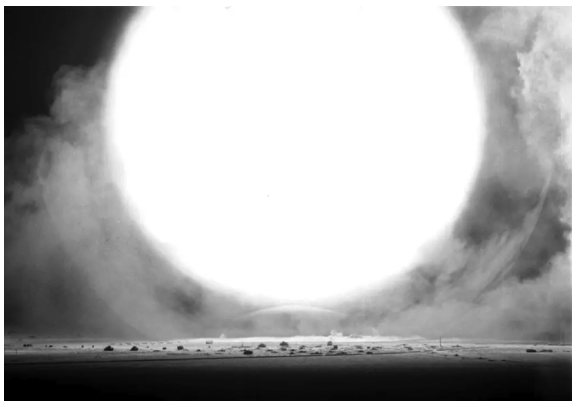


Figure 3.1: The fireball of a nuclear test explosion hangs over the Nevada desert. Though hard to see just below the bottom of the fireball, a crescent-shaped shock wave has bounced off the desert floor and is merging into the expanding nuclear fire (*New York Times*, 14 Sept. 2010).

Apart from E and ρ, the governing parameters of the problem are t, time elapsed after the energy release, and R, the size of the hot region.

At an early stage after the energy release the hot region expands so rapidly that, when the lower part of it encounters the Earth surface (if the explosion charge is located on top of a tower or is dropped from an airplane and explodes in air), the upper part is not affected and remains spherical for a relatively long time. Thus, it is not unreasonable to assume that the expanding hot region is spherically symmetric, hence it is called a fireball.

Applying the dimension matrix. The governing parameters and their dimensions are:

$$[E] = \mathrm{M\,L^2\,T^{-2}}, \qquad [\rho] = \mathrm{M\,L^{-3}}, \qquad [t] = \mathrm{T}, \qquad [R] = \mathrm{L},$$

and our goal is to find how R depends on the other parameters. The dimension matrix has

the form

	E (k_1)	ρ (k_2)	t (k_3)	R (k_4)
M	1	1	0	0
L	2	−3	0	1
T	−2	0	1	0

and equations for k_i follow as

$$\begin{aligned} \mathrm{M}: &\quad k_1 + k_2 = 0, \\ \mathrm{L}: &\quad 2k_1 - 3k_2 + k_4 = 0, \\ \mathrm{T}: &\quad -2k_1 + k_3 = 0, \end{aligned}$$

and we put $k_4 = 1$. This leads to

$$\begin{aligned} \mathrm{M}: &\quad k_2 = -k_1, \\ \mathrm{L}: &\quad 2k_1 - 3k_2 + 1 = 0, \\ \mathrm{T}: &\quad k_3 = 2k_1, \end{aligned}$$

which solves to yield

$$k_1 = -\tfrac{1}{5}, \qquad k_2 = \tfrac{1}{5}, \qquad k_3 = -\tfrac{2}{5}.$$

We have thus derived the following dimensionless complex of the governing parameters:

$$\Pi = E^{-1/5}\,\rho^{1/5}\,t^{-2/5}\,R,$$

which leads to the desired result:

$$\boxed{R = \Pi \left(\frac{Et^2}{\rho}\right)^{1/5}}. \tag{3.3}$$

This is the famous expansion law of an explosion fireball known as the Sedov–Taylor solution. It was first derived in 1941 by Geoffrey Ingram Taylor (1886–1975) in the UK, John von Neumann (1903–1957) in the USA and Leonid Ivanovich Sedov (1907–1999) in the USSR (published in 1946, but derived earlier).

In 1950, Sir Geoffrey Taylor published a paper where he analysed photographs of the first atomic explosion in New Mexico declassified in 1947. Sir Geoffrey found that the expansion of the fireball was described by (3.3) to an impressive degree of accuracy, and obtained, from fitting it to the observed expansion, the explosion energy E. This paper caused a scandal in the USA military and intelligence establishment since, unlike the photographs of the explosion, the charge energy remained highly classified.

This solution also describes explosions of another kind and scale, explosions of young, hot, massive, rapidly evolving stars, leading to the appearance of extremely bright supernova stars. The supernova of 1054 AD was, at its peak, brighter than any star or planet in the sky and remained visible in daylight for over three weeks; it was recorded in historical chronicles around the world (the supernova exploded some 6,500 light years away). Its remnant is now known as the Crab nebula.

Supernova explosions drive the turbulence in the interstellar gas whose scale is close to the size of the fireball at a time when its expansion speed $\mathrm{d}R/\mathrm{d}t$ becomes comparable to the speed of sound in the ambient gas, at the early stages of supersonic expansion no perturbation can propagate ahead of the shock, except for light.