§4.6.1 Bernoulli equations

ODE of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \qquad n \in \mathbb{R}$$

If $n = 0$, solution via integrating factor
$n = 1$, separable

$n \neq 0, 1$ equation is nonlinear, solution via a 'clever' substitution

The substitution: $\div y^n$ (assume $y \neq 0$)

$$y^{-n}y' + P(x)y^{1-n} = Q(x)$$

now introduce $v(x) = y^{1-n}$

chain rule $\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$

hence $\frac{1}{1-n}v' = y^{-n}y'$

substitute into ODE

$$\frac{1}{1-n}v' + P(x)v = Q(x)$$

or $$v' + (1-n)P(x)v = (1-n)Q(x)$$

1st order linear, solve via integrating factor

Eg.

$$\frac{dy}{dx} + \frac{y}{x} = xy^2$$

$P(x) = 1/x$, $Q(x) = x$, $n = 2$

Let $v(x) = y^{1-n} = y^{1-2} = y^{-1} = 1/y$

and $v' = -1/y^2 \, y'$

÷ original ODE by y^2

$$\underbrace{1/y^2 \frac{dy}{dx}}_{-v'} + \underbrace{\frac{1}{y}}_{v} \cdot \frac{1}{x} = x$$

$$-\frac{dv}{dx} + \frac{v}{x} = x$$

or $$\frac{dv}{dx} - \frac{v}{x} = -x$$

1st order linear ODE

Solution via IF

$$\mu(x) = \exp\left(\int -1/x \, dx\right) = 1/x$$

so

$$\frac{d}{dx}\left(v/x\right) = -1 \Rightarrow v(x) = -x^2 + cx$$

now back in terms of $y(x) = 1/v(x)$

$$y = \frac{1}{cx - x^2}$$

(Find c if $y(x=2) = 1/2$)

§4.6.2 Homogeneous 1st order ODEs

Def^n a homogeneous function has the following property:

If we have $f(x_1, x_2, \ldots x_n)$ then if

$$f(sx_1, sx_2, \ldots sx_n) = s^k f(x_1, x_2, \ldots x_n)$$

then f is said to be homogeneous

e.g. $f(x,y) = x^2 + y^2 \Rightarrow f(sx, sy) = s^2 f(x,y)$ ($k=2$)

If

$$\boxed{\frac{dy}{dx} = \frac{M(x,y)}{N(x,y)}}$$

this is said to be a homogeneous ODE if M, N are both homogeneous functions of the same degree (i.e. same k)

Solution method:

$$\frac{dy}{dx} = \frac{M(x,y)}{N(x,y)}$$

such that $M(\lambda x, \lambda y) = \lambda^k M(x,y)$; $N(\lambda x, \lambda y) = \lambda^k N(x,y)$

then let $\lambda = 1/x$ $\quad M(x,y) = \frac{1}{x^k} M(1, y/x)$

$$N(x,y) = \frac{1}{x^k} N(1, y/x)$$

And so we can re-write the ODE

$$\frac{dy}{dx} = \frac{M(x,y)}{N(x,y)} = \frac{x^k}{x^k}\frac{M(1, y/x)}{N(1, y/x)} = F(y/x)$$

Now perform the substitution:

$$\boxed{v = y/x}$$

$$y = vx \quad \therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

or

$$x\frac{dv}{dx} + v = F(v)$$

which we can re-write as

$$x\frac{dv}{dx} = F(v) - v \quad \} \text{ separable}$$

$$\int \frac{dv}{F(v) - v} = \int \frac{dx}{x}$$

always remember to substitute back $v = y/x$

E.g.

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$$

$$M(x,y) = x^2 + y^2 \qquad N(x,y) = 2xy$$

$$M(\lambda x, \lambda y) = \lambda^2 M(x,y) \qquad N(\lambda x, \lambda y) = \lambda^2 N(x,y)$$

Both homogeneous functions of order 2

$$\therefore \quad \frac{dy}{dx} = \frac{1+(y/x)^2}{2(y/x)} = F(y/x)$$

Let $v = y/x$ $\quad \therefore \quad y' = v + xv'$

$$v + xv' = \frac{1+v^2}{2v}$$

or

$$x\frac{dv}{dx} = \frac{1+v^2}{2v} - v$$

$$= \frac{1+v^2-2v^2}{2v} = \frac{1-v^2}{2v}$$

Separable ODE

$$\int \frac{2v}{1-v^2}\,dv = \int \frac{1}{x}\,dx$$

const of $\int$

$$-\log|1-v^2| = \log|x| + \log c$$

$$1-v^2 = c/x$$

now replace $v = y/x$

$$1 - y^2/x^2 = c/x$$

or $\quad \boxed{x^2 - y^2 = cx}$

§ 4.7 2nd order linear ODE (const. coeff)

If $$py'' + qy' + ry = f(x)$$

where p, q, r are constants, then we have a linear 2nd order ODE with const. coefficients

$f(x) = 0$ homogeneous
$f(x) \neq 0$ inhomogeneous

The general solution has the form:

$$y = y_{CF} + y_{PI}$$

y_{PI}: particular integral

y_{CF}: complimentary function

where if we write $L[y] = py'' + qy' + ry$

then $$L[y_{CF}] = 0$$

$$L[y_{PI}] = f(x)$$

so a homogeneous problem only has y_{CF}

§ 4.8 Auxillary equation

$$L = p\frac{d^2}{dx^2} + q\frac{d}{dx} + r$$ ⇐ made up of $\frac{d^{(n)}}{dx^n}$

consider $$L[y] = \lambda y$$

λ: eigenvalue of L

y: eigenfunction of L

If $L = d/dx$

$$\frac{dy}{dx} = \lambda y \Rightarrow y = e^{\lambda x}$$

Eigenfunction of differential operator is an exponential

∴ to solve $py'' + qy' + ry = 0$ make the guess (ansatz)

$$y = e^{mx}$$

$$y' = me^{mx}$$

$$y'' = m^2 e^{mx}$$

∴ after substituting into the ODE we get

$$pm^2 e^{mx} + qme^{mx} + re^{mx} = 0$$

$\div e^{mx} \neq 0$

$$\boxed{pm^2 + qm + r = 0}$$ auxillary equation

a quadratic, we know how to solve those!

Nature of the solution depends on

$$\Delta = q^2 - 4pr \Leftarrow \text{discriminant}$$

Case 1: $\Delta > 0$: 2 real roots m_1, m_2

$$y = Ae^{m_1 x} + Be^{m_2 x}$$

Case 2: $\Delta = 0$: 1 repeated root, m

$$y = (Ax + B)e^{mx}$$

Case 3: $\Delta < 0$: complex conjugate roots $\alpha \pm i\beta$

Euler's formula for $e^{\alpha \pm i\beta}$

$$y = e^{\alpha x}\left(A\cos(\beta x) + B\sin(\beta x)\right)$$

E.g. $y'' - 9y = 0 \Rightarrow m^2 - 9 = 0 \quad m = \pm 3$

$$y = Ae^{3x} + Be^{-3x}$$

see examples in notes.

§4.9 Simple harmonic motion



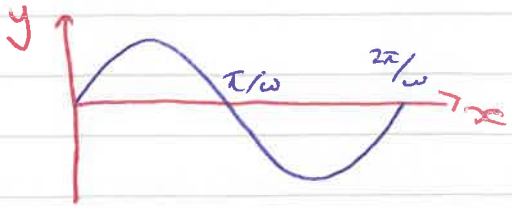
follow equation like

$$\frac{d^2y}{dx^2} + \omega^2 y = 0$$

Auxiliary equation $m^2 + \omega^2 = 0 \Rightarrow m = \pm i\omega$

$$y = A\cos(\omega x) + B\sin(\omega y)$$

or $y = A'\sin(\omega x + \phi)$ ← double angle formula



Damped oscillations

friction / resistance

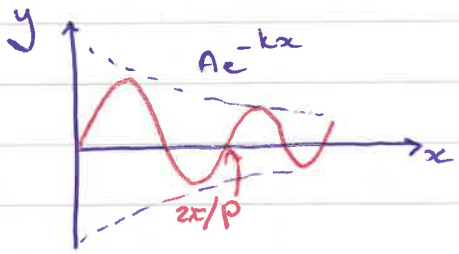
$$y'' + 2ky' + \omega^2 y = 0$$

$$y = e^{mx} \Rightarrow \quad m = -k \pm \sqrt{k^2 - \omega^2}$$

3 cases / regimes:

Underdamped $(k < \omega)$ $\quad m = -k \pm ip$ ($\sqrt{\omega^2 - k^2}$)

$$y(x) = Ae^{-kx} \sin(px + \phi)$$



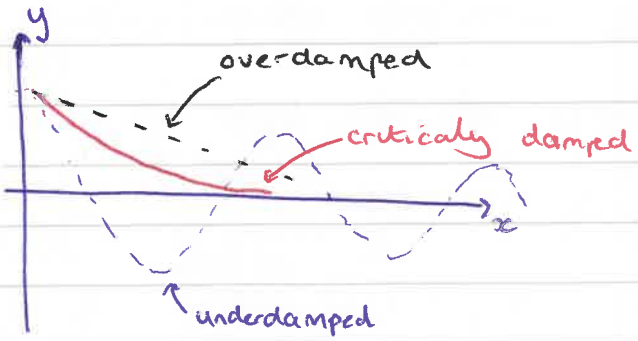
Critically damped $(k = \omega)$

$$y = (A + Bx)e^{-kx}$$

Overdamped $(k > \omega)$ two distinct real roots, m_1, m_2 $\quad m_1, m_2 < 0$

$$y = Ae^{m_1 x} + Be^{m_2 x}$$

sum of decaying exponentials



e.g. $y'' + 3y' + 2y = 6e^{2x}$

① y_{CF}, AE: $m^2 + 3m + 2 = 0 \Rightarrow m = -1, -2$

$$y_{CF} = Ae^{-x} + Be^{-2x}$$

② y_{PI} $f(x) = 6e^{2x}$

$y_{PI} = Ae^{2x}$, $y_{PI}' = 2Ae^{2x}$, & $y_{PI}'' = 4Ae^{2x}$

plug into ODE

$$(4A + 3 \cdot 2A + 2A)e^{2x} = 6e^{2x}$$

$$A = 1/2$$

③ General solution

$$y = Ae^{-x} + Be^{-2x} + 1/2\, e^{2x}$$

e.g. $y'' + 3y' + 2y = 6e^{-x}$

trouble!

$$y_{CF} = Ae^{-x} + Be^{-2x}$$

Try at home $y_{PI} = cxe^{-x}$

Should get $y = -6xe^{-x} + Ae^{-x} + Be^{-2x}$