§4 Ordinary differential equations (ODE)

An ODE is an equation involving an independent variable, e.g. x, t and a dependent variable, e.g. $y(x)$ and at least one of the derivatives e.g. dy/dx

- The order of the differential equation is the highest-order derivative

e.g. $x^2\frac{d^2y}{dx^2} - \sin(x) = 0$ — 2nd order

$\frac{d^4y}{dx^4} - y^5 = 0$ — 4th order

Linearity: an ODE is linear if it can be written

$$a_n(x)\frac{d^n y}{dx^n} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

i.e. no non-linearity of the dependent variable

Differential operators:

function (number) $\Rightarrow$ number

operator (function) $\Rightarrow$ function

e.g. $f(x) = x^2$; $f(z) = 4$

$\mathcal{L}[y] = \frac{d^2y}{dx^2}$; $\mathcal{L}[\sin(x)] = -\sin(x)$

↑ differential operator

Linear operator

An operator is linear if for constants c_1, c_2 and functions $y_1(x), y_2(x)$:

$$L[c_1 y_1 + c_2 y_2] = c_1 L[y_1] + c_2 L[y_2] \quad (*)$$

If L is a linear differential operator

$L[y] = 0$ homogeneous linear ODE

$L[y] = f(x)$ forced / inhomogenous linear ODE

Superposition

If $L[y]_{y_{1,2}} = 0$ then $L[c_1 y_1 + c_2 y_2] = 0$ because of $(*)$

If an ODE is not linear it is called nonlinear

Nonlinear ODEs are typically hard to solve, but are very common!

A solution $y = \phi(x)$ is a solution if the ODE is satisfied for all x in a non-zero interval I.

e.g. $y'' + y = 0 \quad \left(\frac{d^2y}{dx^2} + y = 0\right)$

$y = \sin(x)$ is a solution to the ODE $\forall x$.

So is $y = \cos(x)$ $\therefore$ so is

$$y = A\sin(x) + B\cos(x)$$

This linear combination is called the general solution.

Particular choices of A and B give particular solutions

e.g. $y'' = \frac{d^2y}{dx^2} = 20x^3$ trivial ODE

$\int dx$ twice

$$y = x^5 + c_1 x + c_2$$

The most general solution to an n-th order ODE contains n arbitrary constants.

§4.2 Seperable ODEs

$$\frac{dy}{dx} = g(x)\,h(y)$$

$\int - dx$, note $\frac{dy}{dx} \cdot dx = dy$

$$\int \frac{1}{h(y)} \frac{dy}{dx} \cdot dx = \int g(x)\,dx$$

$$\int \frac{dy}{h(y)} = \int g(x)\,dx$$

E.g. $\frac{dy}{dx} = 4x^3 y^2$

$\therefore \int y^{-2}\,dy = \int 4x^3\,dx$

General sol'n

$$-1/y = x^4 + c$$

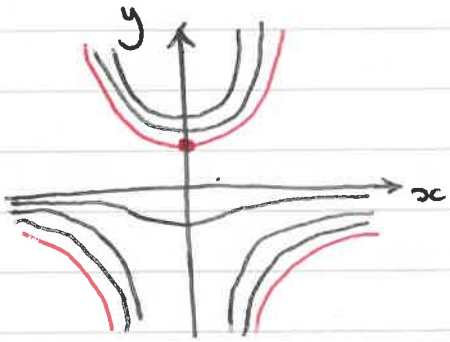
$$y = \frac{-1}{x^4 + c}$$

If we are given additional information e.g.
$y(0) = 1$

$1 = -1/c \qquad \therefore \quad c = -1$

$$y(x) = -\frac{1}{x+1}$$

Particular solution



General solⁿ is a family of curves, additional information pins down which curve

e.g.

$$\frac{dy}{dx} = (6x^2+1)y \qquad y>0$$

$$\int \frac{dy}{y} = \int 6x^2+1 \, dx$$

$$\log y = 2x^3 + x + \ln c$$

$$y = Ce^{2x^3+x}$$

§4.4 Integrating factor

If

$$\frac{dy}{dx} + a(x)y = b(x)$$

We can solve by finding an integrating factor that makes the LHS a total derivative

i.e. $\exists \mu(x)$ such that $\mu\frac{dy}{dx} + \mu a y = \frac{d}{dx}(\mu y)$

why? If we can find a $\mu(x)$ that does this

$$\frac{d}{dx}(\mu y) = \mu b \quad \Leftarrow \text{ easy (?) just } \int$$

can we find this special $\mu(x)$?

$$\mu\frac{dy}{dx} + \mu a y = \frac{d}{dx}(\mu y)$$

product rule

$$= \frac{d\mu}{dx} y + \mu \frac{dy}{dx}$$

$$\boxed{\frac{d\mu}{dx} = \mu a}$$ separable!

$$\int \frac{d\mu}{\mu} = \int a\,dx \Rightarrow \mu = \exp\left\{\int a(x)\,dx\right\}$$

↑ no need to worry about const. any μ will do

and then

$$\boxed{y(x) = \frac{1}{\mu(x)} \int \mu(x) b(x)\,dx}$$

E.g. 4.5

$$xy' - 3y = x^5$$

$\div x$ $\quad y' - \frac{3}{x}y = x^4$ standard form

$$\mu(x) = \exp\left(\int -3/x\,dx\right) = \exp(-3\log x)$$

$$= x^{-3}$$ any constant of $\int$

$\times \mu(x) = x^{-3}$

don't miss x RHS by $\mu(x)$

$$x^{-3}y' - \frac{3}{x^4}y = x$$

$$\frac{d}{dx}\left(x^{-3}y\right) = x$$

normally jump to this line

now integrate $\int dx$:

$$x^{-3}y = x^2/2 + c$$

$\times x^3$

$$\boxed{y = x^5/2 + cx^3}$$

(Find c if $y(1) = 1$)

E.g. $$\frac{dy}{dx} + \frac{x+2}{x}y = e^{-x}$$

$$a(x) = \frac{x+2}{x} = 1 + 2/x$$

$$\therefore \mu(x) = \exp\left(\int 1 + 2/x \, dx\right)$$

$$= e^{x + 2\log x} = x^2 e^x$$

$\times \mu(x) = x^2 e^x$

$$\frac{d}{dx}\left(yx^2e^x\right) = x^2e^x \cdot e^{-x}$$

$$\therefore yx^2e^x = \int x^2 \, dx$$

or $$yx^2e^x = x^3/3 + c$$

$$\boxed{y = e^{-x}\left(x/3 + c/x^2\right)}$$