$$ml^2\ddot{\theta} + b\ell^2\dot{\theta} + mg\ell\sin\theta = 0$$

## §3 The Buckingham-$\pi$ theorem

Lecture 5

In Ex. 2.1 imagine $b=0$ (undamped pendula)

$$\Rightarrow m\ell\ddot{\theta} = -mg\sin\theta$$

$\ddot{\theta} = \frac{d^2\theta}{dt^2}$

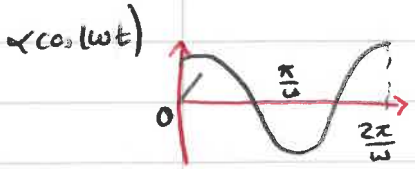
$$\div m\ell \Rightarrow \ddot{\theta} = -\frac{g}{\ell}\sin\theta$$

if $\theta \ll 1$ $\quad \sin\theta \approx \theta$

$$\therefore \ddot{\theta} = -g/\ell\,\theta$$

In §4 we'll learn how to solve this

soln $\theta = \alpha\cos(\omega t)$




angular frequency $\omega = \sqrt{g/\ell}$

the period (time taken for it to complete 1 cyle)

is

$$\tau = 2\pi/\omega = 2\pi\sqrt{\ell/g}$$

Did we need to solve the equation to find this?

Now if we have $\ddot{\theta} = -g/\ell\,\theta$ then the period $\tau$ can only depend on $\ell \propto g$.

Consider the dimensions

$$[\tau] = T \qquad [\ell] = L \qquad [g] = LT^{-2}$$

$$\tau \propto \ell^{\alpha} g^{\beta} \quad \text{must be:} \quad \tau = C\sqrt{\frac{\ell}{g}}$$

full eqn $C = 2\pi$

As the problem becomes more complicated we need a systematic method.

$\Rightarrow$ dimension matrix

Step 1:

Identify the governing parameters, (given in exam)

pendula: $\tau, \ell, g, m$

count them: $n = 4$

Step 2:

Identify their dimensions:

$[\tau] = T \quad [\ell] = L \quad [g] = L/T^2 \quad [m] = M$

how many have independent dimensions? $k = 3$

$$[g] = \frac{[\ell]}{[\tau]^2}$$

The number of independent dimensionless groups is

$$\boxed{m = n - k}$$

Step 3: Dimension 'Matrix' (think of it as a table)

must have:

$(n=1)$ $$\Pi = \tau^{k_1} \ell^{k_2} m^{k_3} g^{k_4}$$

with the exponents chosen so $[\Pi] = 1$

we could 'heuristically' work out $k_1 - k_4$

better to be systematic, dimension matrix (no linear algebra needed):

columns ⇔ parameters  rows ⇔ basic units

e.g. for the pendula:

| | $(k_1)$ $\tau$ | $(k_2)$ $L$ | $(k_3)$ $m$ | $(k_4)$ $g$ |
|---|---|---|---|---|
| M | 0 | 0 | 1 | 0 |
| L | 0 | 1 | 0 | 1 |
| T | 1 | 0 | 0 | −2 |

reading down a column should recover the parameters dimensions

DO THIS CHECK ↗ $[\tau] = T$ ; $[l] = L$ ; $[m] = M$ ; $[g] = LT^{-2}$

Now once we are happy do the rows to make $[\Pi] = 1$:

$$M: \quad 0k_1 + 0k_2 + k_3 + 0\cdot k_4 = 0,$$
$$L: \quad 0k_1 + k_2 + 0k_3 + k_4 = 0,$$
$$T: \quad k_1 + 0k_2 + 0k_3 - 2k_4 = 0,$$

3 equations, 4 unknowns ∴ fix one

OK as just looking for a dimensionless combination

⇒ choose $k_1 = 1$ $(\tau = f(L, m, g))$

equation for m: $k_3 = 0$ (bob mass does not appear)

left with $k_2 + k_4 = 0$,
$k_1 - 2k_4 = 0$
$=1$

$(k_1 = 1)$ $\quad k_4 = 1/2 \;,\; k_2 = -1/2$

$$\therefore \quad \Pi = \tau l^{-1/2} g^{1/2}$$

and setting $\Pi$ = const as $n = 1$

$$\boxed{\tau = \Pi \sqrt{l/g}}$$

Lecture 6

## §3.4 Terminal velocity of a raindrop.




5 governing parameters

$v, m, g, e, A$

but we can simplify at terminal v, weight is balanced by drag $\therefore$ reduce to 4:

$$v, \; W = mg, \; e, \; A$$

$n = 4$ governing parameters, dimensions:

$$[v] = LT^{-1}; \quad [W] = MLT^{-2}; \quad [e] = ML^{-3}; \quad [A] = L^2$$

$k = 3$ base dimensions $(L, M, T)$ and so $m = n - k$

$\therefore$ $m = 1$ a single dimensionless group.

dimension matrix

| | $k_1$ v | $k_2$ W | $k_3$ e | $k_4$ A |
|---|---|---|---|---|
| M | 0 | 1 | 1 | 0 |
| L | 1 | 1 | -3 | 2 |
| T | -1 | -2 | 0 | 0 |

now form equation from rows:

$$M: \quad k_2 + k_3 = 0$$
$$L: \quad k_1 + k_2 - 3k_3 + 2k_4 = 0$$
$$T: \quad -k_1 - 2k_2 = 0$$

we want $V = f(\cdots)$ $\therefore$ $\boxed{k_1 = 1}$

$$T: \quad 2k_2 = -k_1 = -1 \qquad \boxed{k_2 = -1/2}$$

$$M: \quad -1/2 + k_3 = 0 \qquad \boxed{k_3 = 1/2}$$

$$L: \quad 1 - 1/2 - 3/2 + 2k_4 = 0 \qquad \boxed{k_4 = 1/2}$$

and so

$$\Pi = v W^{-1/2} e^{1/2} A^{1/2}$$

$m = 1 \Rightarrow \Pi = \text{const}$

$$\boxed{V = C\sqrt{\frac{W}{eA}} = C\sqrt{\frac{mg}{eA}}}$$

does it make sense? Heavier drops WT fall faster, denser air $e\uparrow$ slows them down.

In the next example we will see how we deal with $m = 2$.

# Buckingham - $\Pi$ theorem

Problem involves $n$ governing parameters with $k$ independent variables

Then we can rewrite the problem in terms of

$$\boxed{m = n - k}$$

dimensionless parameters $\pi_1, \ldots \pi_m$, such that

$$\Phi(\pi_1, \pi_2, \ldots \pi_m) = 0$$

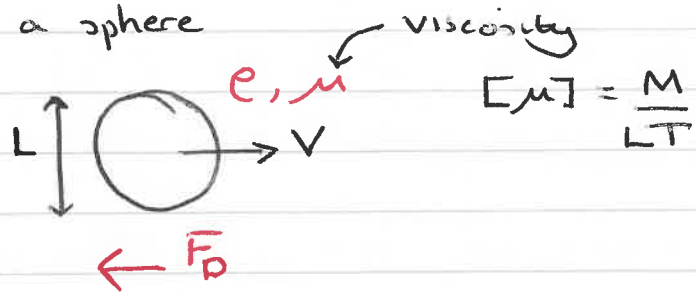
unknown function

If $m = 1$ : $\Phi(\pi_1) = 0$ $\quad \pi_1 = \text{const}$

may be more than 1 const. we don't know how many roots $\Phi$ has.

Proof in notes and not assessed!

§3.5 Drag on a sphere




$[\mu] = \frac{M}{LT}$

1. 5 governing parameters $(n = 5)$

$[F_D] = \frac{ML}{T^2}$ $\quad [L] = L$ $\quad [V] = \frac{L}{T}$ $\quad [\rho] = \frac{M}{L^3}$

$[\mu] = \frac{M}{LT}$

2. 3 independent dimensions $(k = 3)$ : M, L, T

Dimension matrix : looking for $n = 5 - 3 = 2$ groups $\Pi_1, \Pi_2$

| | $k_1$ $F_D$ | $k_2$ L | $k_3$ V | $k_4$ e | $k_5$ $\mu$ |
|---|---|---|---|---|---|
| M | 1 | 0 | 0 | 1 | 1 |
| L | 1 | 1 | 1 | -3 | -1 |
| T | -2 | 0 | -1 | 0 | -1 |

now form simultaneous equations

$$M: \quad k_1 + k_4 + k_5 = 0$$
$$L: \quad k_1 + k_2 + k_3 - 3k_4 - k_5 = 0$$
$$T: \quad -2k_1 - k_3 - k_5 = 0$$

$m = 2$, expecting to choose two $k_i$'s freely

$\Pi_1$ : $k_1 = 1$, $k_5 = 0$

$$M: \quad k_4 = -k_1 = -1$$
$$T: \quad k_3 = -2k_1 = -2$$
$$L: \quad 1 + k_2 - 2 + 3 = 0 \Rightarrow k_2 = -2$$

$$\therefore \quad \Pi_1 = \frac{F_D}{eV^2L^2}$$

$\Pi_2$ : $k_1 = 0$, $k_5 = 1$ ← different choices

M: $k_4 = -1$ ; T: $k_3 = -1$ L: $k_2 = -1$

$$\therefore \quad \Pi_2 = \frac{\mu}{eVL}$$

The theorem tells us

$$\Phi(\pi_1, \pi_2) = 0$$

aside (beyond what you would be asked to show)

$$\pi_1 = \frac{F_D}{\rho V^2 L^2}$$ ⇐ coefficient of drag $C_D$

$$C_D = F\left(\frac{\rho V L}{\mu}\right) = F(Re)$$

↑ Reynolds number