Discrete time models

Lecture 3 - 12/10/26

Imagine you have a population that reproduces once a year. We require a different modelling approach

Difference equation: $x_{n+1} = f(x_n)$

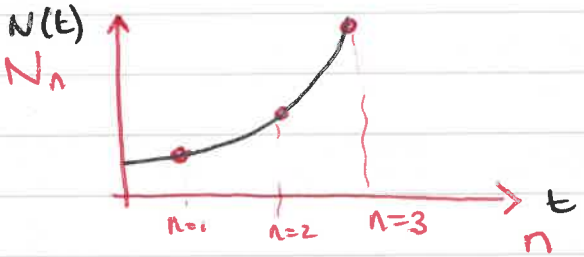
$n = 0, 1, 2 \ldots$

typically we know x_0

We already met $\frac{dN}{dt} = rN$, the discrete analogy is

$$N_{n+1} = RN_n$$

e.g. $N_1 = RN_0$; $N_2 = RN_1 = R^2N_0$ $\quad$ $\boxed{N_n = R^nN_0}$



But similar looking difference and differential equations can also display very different phenomena.

§1.73 First order linear difference equations

$$x_{n+1} = ax_n + b$$



Solution method: seek ~~eqm~~ equilibrium:

$$x_{n+1} = x_n = x^*$$

$$\therefore \quad x^* = ax^* + b \quad \Rightarrow \quad \boxed{x^* = \frac{b}{1-a}}$$

$a \neq 1$

Now we can study the stability of x^*

define $e_n = x_n - x^* \Rightarrow x_n = e_n + x^*$

plug this into $x_{n+1} = ax_n + b \Rightarrow e_{n+1} = ae_n$

But this is just geometric growth

$$e_n = a^n (x_0 - x^*)$$

and so $$x_n = x^* + a^n (x_0 - x^*)$$

Stability: the equilibrium is stable if $|a| < 1$ as $a^n \to 0$. An important (and interesting) case is if $a \in (-1, 0)$ as the solution oscillates.

Example - drug dosing

Patient takes 100mg of a drug each day, and the body eliminates 30% of the drug between doses

$$\therefore D_n = 0.7 D_{n-1} + 100 \qquad D_1 = 100$$

equilibrium is stable

i.e. $a = 0.7$ $b = 100$

$$D^* = \frac{100}{1 - 0.7} \approx 333\text{mg}$$

as $|a| < 1$ we reach D^* irrespective of the starting dose.

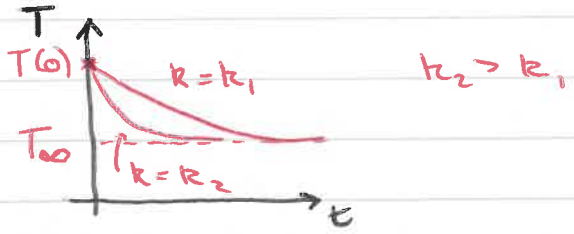


§2 Dimensional analysis & nondimensionalisation

§2.1 Characteristic scales

$$\frac{dT}{dt} = -k(T - T_\infty)$$

Newton's cooling law ⇒ exponential like decay



a characteristic scale is a representative value of a quantity that captures the natural "size" of that quantity in a problem we are analysing

e.g. for T: $T(0) - T_\infty$ the initial 'excess'
for time, t: $1/k$ the natural rate of decay

for length: the size of the domain

Key idea: we can gain insight by making variables dimensionless, i.e. dividing by their characteristic scale.

– EOL –

E.g. 2.3 Newton's cooling revisited

Lecture 4 13/01/26

$$\frac{dT}{dt} = -k(T - T_\infty) \qquad T(0) = T_0$$

① Dimensional variables: T (temperature, K), t (time, s)

k (cooling rate, $1/s$)

② Characteristic scales: temperature $\Delta T = T_0 - T_\infty$

time : $\tau = 1/k$

③ Introduce new non-dimensional parameters rescaling the dependent and independent variables

$$\theta = \frac{T - T_\infty}{T_0 - T_\infty}, \qquad s = kt$$

$[\theta] = [s] = 1$

④ Rewrite: $T = T_\infty + (T_0 - T_\infty)\theta$

$t = s/k$

$$\therefore \quad \frac{dT}{dt} = (T_0 - T_\infty)\frac{d\theta}{ds} \cdot k \qquad \text{LHS}$$

$$-k(T - T_\infty) = -k(T_0 - T_\infty)\theta \qquad \text{RHS}$$

and thus $$\boxed{\frac{d\theta}{ds} = -\theta \; ; \; \theta(0) = 1} \qquad T(0) = T_0$$

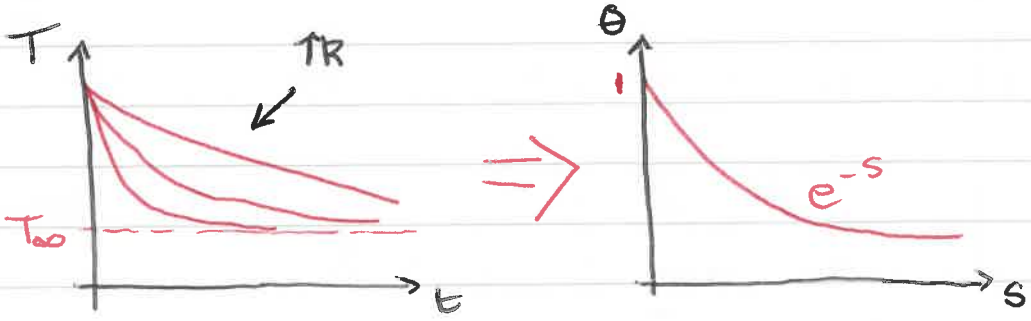
This is a universal equation (no parameters)

solution $$\theta(s) = e^{-s}$$

Every cooling problem maps to this, all solutions collapse to $\theta = e^{-s}$

§2.4 Solution collapse

at the start we sketched



we say the solutions have collapsed to a single curve

At home try the worked example for the logistic equation.

§2.5 Dimensionless groups

When non-dimensionalising more complex equations not all the parameters cancel. What remain are dimensionless groups → they carry physical meaning.

E.g. 2.1

$\frac{d^2\theta}{dt^2} = \ddot{\theta} = \theta''$

$$mL^2\ddot{\theta} + bL^2\dot{\theta} + mgL\sin\theta = 0$$

damped pendulum



θ: dimensionless (radians are a ratio of arc lengths)

$[m]$ = Mass
$[L]$ = Length
$[g] = LT^{-2}$
$[t]$ = Time

b) define $u = \theta$; timescale $\tau = \sqrt{\frac{L}{g}}$ ← [ ] = t

∴ define $s = \frac{t}{\tau}$ ← rescale (if needed) the dependent and independent variables

c) ÷ ODE by mL^2

$$\ddot{\theta} + \frac{b}{m}\dot{\theta} + \frac{g}{L}\sin\theta = 0$$

if $s = \frac{t}{\tau}$ $\quad \dot{\theta} = \frac{1}{\tau}u'$; $\ddot{\theta} = \frac{1}{\tau^2}u''$

∴

$$\frac{1}{\tau^2}u'' + \frac{b}{m}\frac{1}{\tau}u' + \frac{g}{L}\sin u = 0$$

$\times \tau^2 = \frac{L}{g}$

$$u'' + \underbrace{\frac{b}{m}\sqrt{\frac{L}{g}}}_{\gamma}u' + \sin u = 0$$

$\gamma = \frac{b}{m}\sqrt{\frac{L}{g}}$ is dimensionless (check). It quantifies the strength of the dampening

$$u'' + \gamma u' + \sin u = 0$$

at home try exercise 2.2

§3