MATH 4120 Lecture 1 - 05/10/26

§1.1 The modelling cycle

A mathematical model is a set of equations (with parameters), that describes or approximates the behaviour of a real-world (physical, biological) system.

How do we construct them?

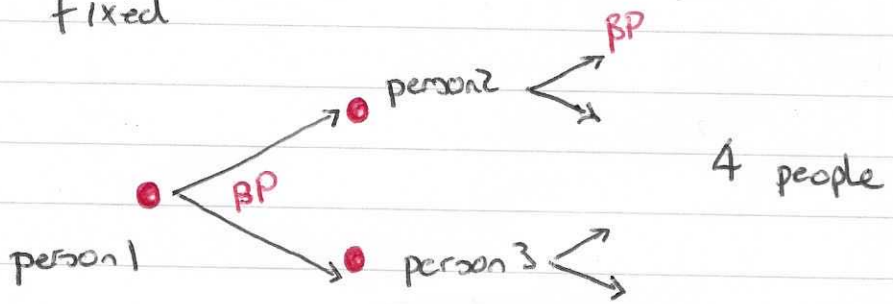
1. Observe
2. Simplify
3. Formulate
4. Solve
5. Interpret
6. Validate
7. Refine

E.g. A rumour starts with one person in a town of 10,000 how long before half of the town has heard it?

① Let's write this mathematically

Let $P = 10{,}000$ and assume P is fixed $\left(\frac{dP}{dt}=0\right)$

assume a) people mix evenly
b) the rate the rumour spreads (β Beta) β is fixed



So a really simple model might be:

$$\frac{dN}{dt} = \beta NP$$

(where $N(t)$ is the number of people that have heard the rumour)

and we would set $N(t=0) = 1$

BUT! Think about this a bit more, this is true when $N \ll P$, as more people know there are less (↑ much less than) people to tell.

So we must account for the fact that the effective rate drops. The next model would be

$$\boxed{\frac{dN}{dt} = \beta N(P-N)}$$ logistic model

units of ↓

$$[\beta] = \frac{1}{\text{person per day}}$$

later in the course we will learn how to solve equations like this and complete the modelling cycle.

After the lecture please read §1.2 the art of simplification

Lecture 2 - 06/10/26

In the first example I introduced the idea of the units of β. Let's explore this more:

Dimensional analysis is a simple, but powerful, way to check a model is valid and gain insight

§1.3 Units and dimensions

Every physical quantity has dimensions (the type of measurement it represents) and is expressed in a particular units. For example:

Dimension	Symbol	Unit (SI)	Unit (imperial)
Length	L	metre (m)	feet
Mass	M	kilogram (kg)	pounds
Time	T	seconds (s)	seconds
Temperature	Θ (capital theta)	kelvin (K)	farenheit

We write $[Q]$ for the dimensions of a quantity Q.

e.g. $$[\text{speed}] = \left[\frac{\text{distance}}{\text{time}}\right] = \frac{[\text{distance}]}{[\text{time}]}$$

$$= \frac{L}{T} = \boxed{LT^{-1}}$$

$$[\text{acceleration}] = \frac{[\text{speed}]}{[\text{time}]} = \frac{L}{T^2} = \boxed{LT^{-2}}$$

$$[\text{force}] = [\text{mass}] \times [\text{acceleration}]$$

$$= \boxed{MLT^{-2}}$$

and we can now see how useful this is when we seek:

$$[\text{energy}]$$

$E_{kinetic} = 1/2 \times \text{mass} \times \text{velocity}^2 = 1/2\, mv^2$

$E_{work} = \text{force} \times \text{distance} = F \times L$

$\therefore$ $[\text{Energy}] = [1/2] \times [\text{mass}] \times [\text{velocity}]^2$ (1)

and $[\text{Energy}] = [\text{Force}] \times [\text{distance}]$ (2)

(1): $1 \times M \times (L/T)^2 = ML^2/T^2$ ✓

(2): $M \times \frac{L}{T^2} \times L = ML^2/T^2$ ✓ OK

But this idea also applies to each term in an equation (including differential ones)

> Every term in an equation must have the same dimensions or the equation is not valid.

E.g. a claim is made that the distance fallen under gravity is

$$s = 1/2\, g^2 t \qquad [s] = L \qquad [g] = \frac{L}{T^2}$$

check: $L \neq \left(\frac{L}{T^2}\right)^2 \cdot T = L^2/T^3$!!

Instead we can see the power of this approach by not even solving the equation if we know the only relevant variables are g, t so we simply assume

$$s \propto g^\alpha t^\beta$$

$$\therefore \left(\frac{L}{T^2}\right)^{\alpha} \cdot T^{\beta} = L \Rightarrow \alpha = 1, \beta = 2$$

$$S \propto gt^2 \quad \text{or} \quad S = Cgt^2$$

constant

solving full equation

$$\boxed{\frac{d^2S}{dt^2} = g}$$

$\left(S(0), \frac{dS}{dt}(0) \text{ gives } C = \frac{1}{2}\right)$

Try yourself with Newton's law of gravity

$$F = \frac{G m_1 m_2}{r^2}$$

Separation of variables

Before we go any further we need to be able to solve (step 4) a differential equation. We will now introduce the first technique: separation of variables.

THE IDEA: If we can write a differential equation in the form

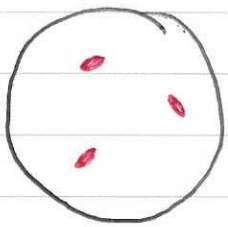
$$\frac{dy}{dx} = \underbrace{g(x)}_{\text{function of } x}\, \underbrace{h(y)}_{\text{function of } y}$$

then

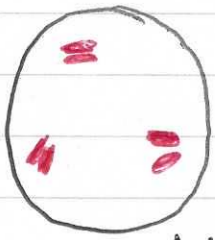
$$\int \frac{1}{h(y)}\, dy = \int g(x)\, dx$$

Why? We'll discuss later.

§1.4 Exponential growth



$t=0$



$t=1/r$

find population has doubled over period $1/r$ $\quad [r] = \frac{1}{T}$

$r_1 N \neq r_2 N$

Simplest model $\quad \frac{dN}{dt} = rN \qquad N(0) = N_0 \qquad [N] = 1$

check dimensions $\quad \left[\frac{dN}{dt}\right] = \frac{[dN]}{[dt]} = \frac{1}{T}$

now solve:

$$\frac{dN}{dt} = rN \implies \int \frac{1}{N} dN = \int r\, dt$$

$$\log|N| = rt + \ln C \qquad \text{const. of } \int$$

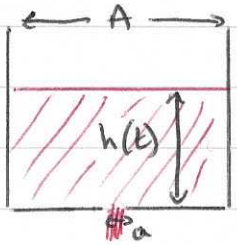
C

exponentiate both sides: $\quad N = N_0 e^{rt}$

and so we predict the population grows exponentially

true at first, eventually system size / resources limit growth.

§1.5 Draining tank example



Volume must be

$$V = Ah(t)$$

call the outflow rate $Q \qquad \frac{dV}{dt} = A\frac{dh}{dt} = -Q$

Torricelli's law $v(t) = \sqrt{2gh}$ ← outflow speed

$Q = aV = a\sqrt{2gh}$, and thus,

$$\frac{dV}{dt} = \boxed{A\frac{dh}{dt} = -a\sqrt{2gh}}$$

suggestion, check this is dimensionally consistent

Now solve, separable equation. For ease call

$$c = \frac{a}{A}\sqrt{2g} \quad \therefore \quad \frac{dh}{dt} = -ch^{1/2}$$

or

$$\int h^{-1/2}\,dh = \int -c\,dt$$

$$2\sqrt{h} = C - ct \quad \leftarrow \text{const of } \int$$

If at $t=0$ $h=h_0$ find $C = 2\sqrt{h_0}$

$$h(t) = \left(\sqrt{h_0} - \frac{ct}{2}\right)^2$$

$$\boxed{h(t) = h_0\left(1 - \frac{t}{t_{drain}}\right)^2}$$

$$t_{drain} = \frac{A}{a}\sqrt{\frac{2h_0}{g}}$$

At home work through §1.6 and the drug concentration example