MATH4120: Mathematical Modelling and Programming

Mid-semester test: specimen paper

Chapters 1–4: modelling, dimensional analysis, the Π-theorem, and ordinary differential equations

SPECIMEN PAPER
For practice

Instructions. This specimen paper has the same format and level of difficulty as the mid-semester test. Answer **all five** questions. The paper carries **50 marks**, and the marks for each part are shown in brackets. Please show all your working: marks are given for method as well as for final answers.

Question 1 [10 marks]

A library tracks the number of books out on loan. Let B_n be the number of books on loan at the start of week n. During each week, 600 books are borrowed, and then, by the end of the week, two thirds of all the books on loan are returned.

(a) Write down the difference equation for B_n in the standard form $B_{n+1} = aB_n + b$, and explain what a and b represent. **[2]**

(b) Find the equilibrium B^* and determine whether it is stable or unstable, justifying your answer. **[3]**

(c) Write down the general solution, and hence find B_n when the library opens with no books on loan, $B_0 = 0$. **[2]**

(d) Find the smallest n for which B_n is within 1% of the equilibrium. **[3]**

Question 2 [10 marks]

A pollutant is released into a lake at a constant rate $P > 0$. Some of it is removed when pairs of pollutant molecules react with each other, at a rate kC^2 ($k > 0$), and some is flushed out by the river, at a rate λC. The concentration $C(t)$ therefore satisfies

$$\frac{\mathrm{d}C}{\mathrm{d}t} = P - kC^2 - \lambda C, \qquad C(0) = 0.$$

Write N for the dimension of concentration.

(a) Find the dimensions of P, k and λ. **[2]**

(b) Show that $C_s = \sqrt{P/k}$ has the dimensions of concentration and that $T_s = 1/\sqrt{Pk}$ has the dimensions of time. **[2]**

(c) Let $c = C/C_s$ and $s = t/T_s$. Use the chain rule to show that

$$\frac{\mathrm{d}c}{\mathrm{d}s} = 1 - c^2 - \alpha c, \qquad c(0) = 0,$$

and find the dimensionless group α in terms of P, k and λ. **[4]**

(d) Find the positive equilibrium c^* of the dimensionless equation. What is c^* when $\alpha = 0$, and what does this mean physically? **[2]**

Question 3 [10 marks]

A ball is kicked from flat ground with speed V, at a fixed launch angle. The horizontal distance R it travels before landing depends on V, the gravitational acceleration g and the mass m of the ball. Ignore air resistance.

(a) Write down the dimensions of R, V, g and m. **[2]**

(b) Construct the dimension matrix, state how many dimensionless groups there are, and solve for the exponents to show that

$$R = C\,\frac{V^2}{g}$$

for some dimensionless constant C. Explain why the mass cannot appear in the answer. **[4]**

(c) A ball kicked at 20 $\mathrm{m\,s^{-1}}$ lands 30 m away. Estimate how far it would travel if kicked at 10 $\mathrm{m\,s^{-1}}$ at the same angle. **[2]**

(d) In reality R also depends on the launch angle θ (in radians). How many dimensionless groups are there now? Write down the form of the result. **[2]**

Question 4 [10 marks]

(a) Use the substitution $v = 1/y$ to solve the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y = xy^2, \qquad y(0) = 1.$$

[4]

(b) Find the complementary function of

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = 6e^{2x}.$$

[2]

(c) Find a particular integral, and hence the general solution, of the equation in (b). **[2]**

(d) Find the solution of the equation in (b) that satisfies $y(0) = 0$ and $y'(0) = 0$. **[2]**

Question 5 [10 marks]

A population of deer on an island breeds once a year, and then some of the deer are culled. Let N_n be the number of deer at the start of year n. The code below computes N_n for $n = 0, 1, \ldots, 20$, starting from $N_0 = 400$.

```
1  R = 2.0      # growth factor per year
2  K = 1000     # carrying capacity
3  H = 50       # deer culled each year
4
5  def next_year(N):
6      return R * N * (1 - N / K) - H
7
8  N = np.zeros(21)
9  N[0] = 400
10 for n in range(20):
11     N[n + 1] = next_year(N[n])
```

In the code, `np.zeros(21)` is an array of 21 zeros, and `range(20)` gives $n = 0, 1, \ldots, 19$.

(a) Which difference equation does the code carry out? Say what `R`, `K` and `H` stand for. **[2]**

(b) What value does `N[1]` hold, and what does it mean? **[2]**

(c) Suppose instead that the deer are culled before they breed. Write down the new line 6. **[3]**

(d) Suppose instead that the cull grows by 10 every year: 50 deer in year 0, 60 in year 1, and so on. Which lines of the code need to change, and how? **[3]**