MATH4120: Mathematical Modelling and Programming

Mid-semester test: specimen paper

Chapters 1–4: modelling, dimensional analysis, the Π-theorem, and ordinary differential equations

SPECIMEN PAPER
For practice

Model solutions, do not circulate until students have attempted the paper.

Instructions. This specimen paper has the same format and level of difficulty as the mid-semester test. Answer **all five** questions. The paper carries **50 marks**, and the marks for each part are shown in brackets. Please show all your working: marks are given for method as well as for final answers.

Question 1

[10 marks]

A library tracks the number of books out on loan. Let B_n be the number of books on loan at the start of week n. During each week, 600 books are borrowed, and then, by the end of the week, two thirds of all the books on loan are returned.

(a) Write down the difference equation for B_n in the standard form $B_{n+1} = aB_n + b$, and explain what a and b represent. **[2]**

(b) Find the equilibrium B^* and determine whether it is stable or unstable, justifying your answer. **[3]**

(c) Write down the general solution, and hence find B_n when the library opens with no books on loan, $B_0 = 0$. **[2]**

(d) Find the smallest n for which B_n is within 1% of the equilibrium. **[3]**

Solution

(a) Start the week with B_n books on loan. After 600 are borrowed there are $B_n + 600$; two thirds of these are returned, leaving one third. So

$$B_{n+1} = \tfrac{1}{3}(B_n + 600) = \tfrac{1}{3}B_n + 200,$$

which is the standard form with $a = \frac{1}{3}$ and $b = 200$. The factor a is the fraction of books on loan that are *not* returned during the week, and $b = 200$ is the number of the week's 600 new loans that are still out at the end of it. **[2]**

Marking: 1 for the correct difference equation, 1 for identifying and interpreting a and b. The order matters: returning first and then borrowing gives $B_{n+1} = \frac{1}{3}B_n + 600$. Award 1 for this, and follow through in (b)–(d).

(b) The equilibrium satisfies $B^* = aB^* + b$, so

$$B^* = \frac{b}{1-a} = \frac{200}{2/3} = 300.$$

It is stable, because $|a| = \frac{1}{3} < 1$: the distance from equilibrium shrinks by a factor of 3 each week. **[3]**

Marking: 1 for the method, 1 for $B^* = 300$, 1 for "stable" with the reason $|a| < 1$.
(c) $B_n = B^* + a^n(B_0 - B^*)$, so with $B_0 = 0$,

$$B_n = 300 + \left(\tfrac{1}{3}\right)^n (0 - 300) = 300\left(1 - \frac{1}{3^n}\right).$$

[2]

Marking: 1 for the general form, 1 for the particular solution.
(d) 1% of 300 is 3, so we need $300 - B_n = 300/3^n \le 3$, i.e. $3^n \ge 100$. Since $3^4 = 81 < 100$ and $3^5 = 243 \ge 100$, the smallest such n is $n = 5$. (Check: $B_4 \approx 296.3$ and $B_5 \approx 298.8$.) **[3]**
Marking: 1 for setting up the inequality, 1 for $3^n \ge 100$ (or equivalent), 1 for $n = 5$. Accept a correct answer found by listing $B_1, B_2, \ldots$ with working shown.

Question 2

[10 marks]

A pollutant is released into a lake at a constant rate $P > 0$. Some of it is removed when pairs of pollutant molecules react with each other, at a rate kC^2 ($k > 0$), and some is flushed out by the river, at a rate λC. The concentration $C(t)$ therefore satisfies

$$\frac{\mathrm{d}C}{\mathrm{d}t} = P - kC^2 - \lambda C, \qquad C(0) = 0.$$

Write N for the dimension of concentration.

(a) Find the dimensions of P, k and λ. **[2]**

(b) Show that $C_s = \sqrt{P/k}$ has the dimensions of concentration and that $T_s = 1/\sqrt{Pk}$ has the dimensions of time. **[2]**

(c) Let $c = C/C_s$ and $s = t/T_s$. Use the chain rule to show that

$$\frac{\mathrm{d}c}{\mathrm{d}s} = 1 - c^2 - \alpha c, \qquad c(0) = 0,$$

and find the dimensionless group α in terms of P, k and λ. **[4]**

(d) Find the positive equilibrium c^* of the dimensionless equation. What is c^* when $\alpha = 0$, and what does this mean physically? **[2]**

Solution

(a) Every term has the dimensions of $\mathrm{d}C/\mathrm{d}t$, i.e. $\mathrm{N\,T^{-1}}$. So $[P] = \mathrm{N\,T^{-1}}$. From $[k]\,\mathrm{N}^2 = \mathrm{N\,T^{-1}}$, $[k] = \mathrm{N^{-1}T^{-1}}$. From $[\lambda]\,\mathrm{N} = \mathrm{N\,T^{-1}}$, $[\lambda] = \mathrm{T^{-1}}$. **[2]**
Marking: 1 for $[P]$ and $[\lambda]$, 1 for $[k]$.
(b)

$$\left[\frac{P}{k}\right] = \frac{\mathrm{N\,T^{-1}}}{\mathrm{N^{-1}T^{-1}}} = \mathrm{N}^2, \quad \text{so } [C_s] = \mathrm{N}; \qquad [Pk] = \mathrm{N\,T^{-1}} \cdot \mathrm{N^{-1}T^{-1}} = \mathrm{T^{-2}}, \quad \text{so } [T_s] = \mathrm{T}.$$

[2]

Marking: 1 for each.
(c) $C = C_s c$ and $t = T_s s$. By the chain rule,

$$\frac{\mathrm{d}C}{\mathrm{d}t} = \frac{\mathrm{d}C}{\mathrm{d}s} \cdot \frac{\mathrm{d}s}{\mathrm{d}t} = C_s \frac{\mathrm{d}c}{\mathrm{d}s} \cdot \frac{1}{T_s} = \sqrt{\frac{P}{k}}\sqrt{Pk}\,\frac{\mathrm{d}c}{\mathrm{d}s} = P\,\frac{\mathrm{d}c}{\mathrm{d}s}.$$

The right-hand side is

$$P - k\frac{P}{k}c^2 - \lambda\sqrt{\frac{P}{k}}\,c = P - Pc^2 - \lambda\sqrt{\frac{P}{k}}\,c.$$

Dividing by P,

$$\frac{\mathrm{d}c}{\mathrm{d}s} = 1 - c^2 - \frac{\lambda}{\sqrt{Pk}}\,c, \qquad \text{so} \qquad \alpha = \frac{\lambda}{\sqrt{Pk}}.$$

At $t = 0$, $s = 0$ and $C = 0$, so $c(0) = 0$. **[4]**

Marking: 1 for the chain rule with the factor $1/T_s$, 1 for $\mathrm{d}C/\mathrm{d}t = P\,\mathrm{d}c/\mathrm{d}s$, 1 for the right-hand side, 1 for α (and the initial condition).

(d) Setting $\mathrm{d}c/\mathrm{d}s = 0$ gives $c^2 + \alpha c - 1 = 0$, whose positive root is

$$c^* = \frac{-\alpha + \sqrt{\alpha^2 + 4}}{2}.$$

When $\alpha = 0$ (no flushing), $c^* = 1$, i.e. $C^* = C_s = \sqrt{P/k}$: the concentration at which the reaction alone removes pollutant as fast as it is released. (For $\alpha > 0$, $c^* < 1$: flushing lowers the steady concentration.) **[2]**

Marking: 1 for c^*, 1 for $c^* = 1$ with an interpretation.

Question 3

[10 marks]

A ball is kicked from flat ground with speed V, at a fixed launch angle. The horizontal distance R it travels before landing depends on V, the gravitational acceleration g and the mass m of the ball. Ignore air resistance.

(a) Write down the dimensions of R, V, g and m. **[2]**

(b) Construct the dimension matrix, state how many dimensionless groups there are, and solve for the exponents to show that

$$R = C\,\frac{V^2}{g}$$

for some dimensionless constant C. Explain why the mass cannot appear in the answer. **[4]**

(c) A ball kicked at 20 $\mathrm{m\,s^{-1}}$ lands 30 m away. Estimate how far it would travel if kicked at 10 $\mathrm{m\,s^{-1}}$ at the same angle. **[2]**

(d) In reality R also depends on the launch angle θ (in radians). How many dimensionless groups are there now? Write down the form of the result. **[2]**

Solution

(a) $[R] = \mathrm{L}$, $[V] = \mathrm{L\,T^{-1}}$, $[g] = \mathrm{L\,T^{-2}}$, $[m] = \mathrm{M}$. **[2]**

Marking: 1 for $[V]$ and $[g]$, 1 for $[R]$ and $[m]$.

(b) With $\Pi = R^{k_1}V^{k_2}g^{k_3}m^{k_4}$:

	R (k_1)	V (k_2)	g (k_3)	m (k_4)
M	0	0	0	1
L	1	1	1	0
T	0	−1	−2	0

Here $n = 4$ and $k = 3$, so there is $4 - 3 = 1$ group. The rows give $k_4 = 0$, $k_1 + k_2 + k_3 = 0$ and $-k_2 - 2k_3 = 0$. With $k_1 = 1$: $k_2 = -2k_3$, so $1 - 2k_3 + k_3 = 0$, giving $k_3 = 1$ and $k_2 = -2$.

So $\Pi = Rg/V^2$, and $\Pi = C$ gives $R = CV^2/g$. The mass is the only parameter involving M, so nothing can cancel its dimension: the M row forces $k_4 = 0$. **[4]**

Marking: 1 for the matrix, 1 for one group, 1 for the exponents and result, 1 for explaining why m drops out.

(c) $R \propto V^2$, so halving the speed divides the range by 4: about $30/4 = 7.5\,\text{m}$. **[2]**

Marking: 1 for using $R \propto V^2$, 1 for 7.5 m.

(d) θ is already dimensionless, so now $n = 5$ and $k = 3$, giving 2 groups: Rg/V^2 and θ itself. The result becomes

$$R = \frac{V^2}{g} F(\theta)$$

for some unknown function F. (Solving the equations of motion gives $F(\theta) = \sin 2\theta$.) **[2]**

Marking: 1 for two groups with θ as the second, 1 for the form of the result.

Question 4

[10 marks]

(a) Use the substitution $v = 1/y$ to solve the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y = xy^2, \qquad y(0) = 1.$$

[4]

(b) Find the complementary function of

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = 6e^{2x}.$$

[2]

(c) Find a particular integral, and hence the general solution, of the equation in (b). **[2]**

(d) Find the solution of the equation in (b) that satisfies $y(0) = 0$ and $y'(0) = 0$. **[2]**

Solution

(a) This is a Bernoulli equation with $P = 1$, $Q = x$, $n = 2$. With $v = y^{-1}$, $v' = -y^{-2}y'$. Dividing the equation by y^2: $y^{-2}y' + y^{-1} = x$, i.e. $-v' + v = x$, so

$$v' - v = -x.$$

The integrating factor is e^{-x}:

$$\frac{\mathrm{d}}{\mathrm{d}x}(e^{-x}v) = -xe^{-x} \quad \Longrightarrow \quad e^{-x}v = \int -xe^{-x}\,\mathrm{d}x = (x+1)e^{-x} + C,$$

integrating by parts. So $v = x + 1 + Ce^x$. At $x = 0$, $v = 1/y = 1$, so $1 = 1 + C$ and $C = 0$. Hence

$$y = \frac{1}{x+1}$$

(valid for $x > -1$). **[4]**

Marking: 1 for the linear equation for v, 1 for the integrating factor, 1 for $v = x + 1 + Ce^x$, 1 for the final answer.

(b) The auxiliary equation is $m^2 - 4m + 3 = (m-1)(m-3) = 0$, with roots 1 and 3, so the complementary function is $c_1e^x + c_2e^{3x}$. **[2]**

Marking: 1 for the roots, 1 for the complementary function.

(c) Since 2 is not a root, try $\phi = Ae^{2x}$, so $\phi' = 2Ae^{2x}$ and $\phi'' = 4Ae^{2x}$:

$$(4 - 8 + 3)Ae^{2x} = 6e^{2x} \quad \Longrightarrow \quad -A = 6, \quad A = -6.$$

The general solution is

$$y = c_1 e^x + c_2 e^{3x} - 6e^{2x}.$$

[2]

Marking: 1 for the trial form and finding A, 1 for the general solution.

(d) $y(0) = 0$ gives $c_1 + c_2 - 6 = 0$. Since $y' = c_1 e^x + 3c_2 e^{3x} - 12e^{2x}$, $y'(0) = 0$ gives $c_1 + 3c_2 - 12 = 0$. Subtracting, $2c_2 = 6$, so $c_2 = 3$ and $c_1 = 3$:

$$y = 3e^x + 3e^{3x} - 6e^{2x} = 3e^x\left(e^x - 1\right)^2.$$

[2]

Marking: 1 for the two equations, 1 for the correct constants. Follow through from an incorrect general solution.

Question 5 [10 marks]

A population of deer on an island breeds once a year, and then some of the deer are culled. Let N_n be the number of deer at the start of year n. The code below computes N_n for $n = 0, 1, \ldots, 20$, starting from $N_0 = 400$.

```
1  R = 2.0      # growth factor per year
2  K = 1000     # carrying capacity
3  H = 50       # deer culled each year
4
5  def next_year(N):
6      return R * N * (1 - N / K) - H
7
8  N = np.zeros(21)
9  N[0] = 400
10 for n in range(20):
11     N[n + 1] = next_year(N[n])
```

In the code, `np.zeros(21)` is an array of 21 zeros, and `range(20)` gives $n = 0, 1, \ldots, 19$.

(a) Which difference equation does the code carry out? Say what `R`, `K` and `H` stand for. **[2]**

(b) What value does `N[1]` hold, and what does it mean? **[2]**

(c) Suppose instead that the deer are culled before they breed. Write down the new line 6. **[3]**

(d) Suppose instead that the cull grows by 10 every year: 50 deer in year 0, 60 in year 1, and so on. Which lines of the code need to change, and how? **[3]**

Solution

(a) Line 11 sets N_{n+1} to `next_year(N[n])`, so

$$N_{n+1} = R\,N_n\Big(1 - \frac{N_n}{K}\Big) - H.$$

`R` is the growth factor per year (for a small population), `K` the carrying capacity, and `H` the number of deer culled each year. **[2]**

Marking: 1 for the difference equation, 1 for the three meanings.

(b) With $n = 0$, `N[1]` $= 2 \times 400 \times (1 - 400/1000) - 50 = 480 - 50 = 430$. There are 430 deer at the start of year 1. **[2]**

Marking: 1 for 430, 1 for the meaning.

(c) The cull now comes first, so the deer that breed are the $N - H$ left after it:

```
return R * (N - H) * (1 - (N - H) / K)
```

[3]

Marking: 1 for $N - H$ in place of N, 1 for using it in both places, 1 for removing the final $-H$.

(d) `next_year` is only given N, so it cannot tell which year it is. Give it the year as a second input, and use it in the cull:

```
def next_year(N, n):
    return R * N * (1 - N / K) - (50 + 10 * n)
```

and change line 11 to `N[n + 1] = next_year(N[n], n)`. So lines 5, 6 and 11 change.

Another correct answer: leave `next_year` as it is and add `H = 50 + 10 * n` inside the loop, before line 11 (`next_year` uses the current value of `H`). **[3]**

Marking: 2 for a correct change, in either way (1 if the new cull is right but `next_year` *is not given the year); 1 for the reason.*