MATH4120: Mathematical Modelling and Programming

Workshop sheet 5: Single species models

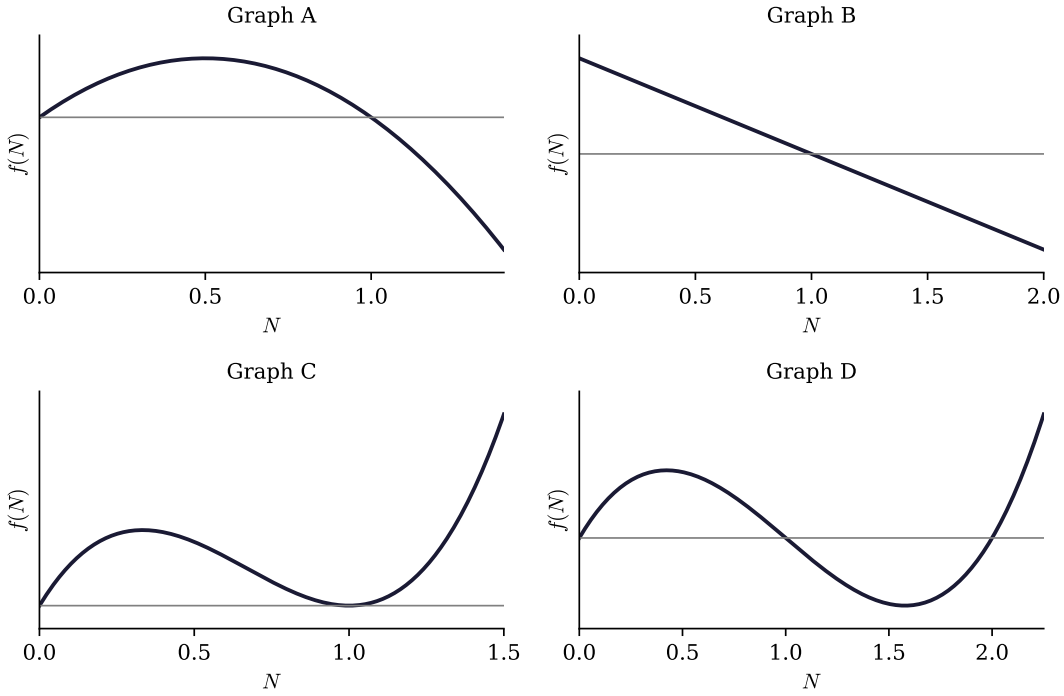
Chapter 5: phase lines, stability and linearisation, harvesting and bifurcations, and the Gompertz model

NOT ASSESSED
Week 7 workshops

This sheet is not assessed. It accompanies the Week 7 workshop and gives further practice with the Chapter 5 material, alongside Exercises 5.1–5.5 in the notes. Question 1 is designed for discussion in small groups, so start there. Question 5 is optional, for anyone who finishes early. Parts marked **[Extension]** are more challenging: try them once you are comfortable with the rest of the sheet.

Question 1

Group discussion: reading the phase line. Each graph below shows the right-hand side $f(N)$ of an autonomous model $\mathrm{d}N/\mathrm{d}t = f(N)$, for $N \geq 0$. No formulas are given: work only from the pictures.



(a) For each graph, draw the phase line: mark every equilibrium and add arrows showing whether N increases or decreases. Classify each equilibrium as stable or unstable.

(b) Match three of the graphs to these descriptions:

(i) A population that grows when small and levels off at a carrying capacity.

(ii) Fish in a lake that is topped up by a hatchery at a constant rate, while a fixed fraction of the fish die each year.

(iii) An insect pest kept at a low level by birds. If the pest ever exceeds a threshold, the birds cannot keep up and the pest population explodes.

(c) In the remaining graph, one equilibrium is neither stable nor unstable. Describe what happens to populations starting just below it and just above it. What does the linearised stability test of Section 5.6 say about this equilibrium, and why?

Question 2

Linearisation: the pendulum in honey. In Question 1 of Workshop sheet 2 you found that a pendulum swinging in honey creeps back towards the vertical, because damping and gravity balance and inertia is negligible. With time measured in units of the creep time $bL/(mg)$, the angle $\theta(s)$ from the downward vertical satisfies

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = -\sin\theta.$$

(a) Find all the equilibria. What do $\theta = 0$ and $\theta = \pi$ correspond to physically?

(b) Use $f'(\theta)$ to classify each equilibrium as stable or unstable. Does your answer make physical sense?

(c) Sketch the phase line for $-2\pi \le \theta \le 2\pi$.

(d) The pendulum is released from rest at $\theta = 3$ radians, just short of upside down. Describe what happens. Why is the motion very slow at first?

(e) **[Extension]** Solve the equation exactly by separating the variables, using the integral

$$\int \frac{\mathrm{d}\theta}{\sin\theta} = \log\left|\tan\frac{\theta}{2}\right| + C,$$

and show that $\tan(\theta/2) = \tan(\theta_0/2)\, e^{-s}$, where $\theta_0 = \theta(0)$.

Question 3

Harvesting a fixed fraction instead of a fixed quota. Section 5.7 mentions *proportional* harvesting, $H = H_1 N$, in which a fixed fraction of the population is removed. For the logistic model,

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \alpha N\left(1 - \frac{N}{M}\right) - H_1 N,$$

where α, M and H_1 are positive constants.

(a) Find the equilibria. For which values of H_1 is there a positive equilibrium?

(b) Use linearisation to find the stability of each equilibrium, for $H_1 < \alpha$ and for $H_1 > \alpha$.

(c) As in Section 5.11, write $N = Mn$ and $t = \tau/\alpha$. Show that

$$\frac{\mathrm{d}n}{\mathrm{d}\tau} = n(1-n) - hn, \qquad h = \frac{H_1}{\alpha},$$

and sketch the bifurcation diagram (n against h), using solid lines for stable equilibria and dashed lines for unstable ones. How does it differ from Figure 5.1?

(d) At equilibrium the harvest per unit time is the *sustainable yield* $Y = H_1 N^*$. Find the value of H_1 that maximises Y, and the maximum yield.

(e) Suppose the fishery is managed by setting a value of H_1 (a fraction) instead of a value of H (a quota). Use your bifurcation diagram, and Figure 5.1, to explain why this is safer if α has been overestimated.

Question 4

The Gompertz model. Tumours, and some animal and plant populations, are often modelled by the *Gompertz* equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \alpha N \log\left(\frac{M}{N}\right), \qquad N(0) = N_0 > 0,$$

where $\alpha > 0$ and $M > 0$ are constants.

(a) Show that $N = M$ is an equilibrium, and use linearisation to show that it is stable.

(b) As $N \to 0^+$, $N \log(M/N) \to 0$, so $N = 0$ also behaves like an equilibrium. Find $f'(N)$, and explain why the linearised stability test cannot be applied at $N = 0$. Use the sign of f to decide whether $N = 0$ is stable.

(c) Let $u = \log(N/M)$. Use the chain rule to show that $\mathrm{d}u/\mathrm{d}t = -\alpha u$. Solve for u, and hence show that

$$N(t) = M \exp\Big(\log\left(\tfrac{N_0}{M}\right) e^{-\alpha t}\Big).$$

(d) At what population is the growth rate $\mathrm{d}N/\mathrm{d}t$ largest? Compare with the logistic model, for which the answer is $M/2$.

(e) Compare the per-capita growth rates $\frac{1}{N}\frac{\mathrm{d}N}{\mathrm{d}t}$ of the Gompertz and logistic models as $N \to 0$. Which model grows faster when the population is very small?

Question 5

(Optional) Growth that slows down. Instead of a crowding term, suppose the growth rate itself decays over time, perhaps because a resource is gradually used up:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \alpha_0 e^{-at} N, \qquad N(0) = N_0,$$

where α_0 and a are positive constants.

(a) Solve this separable equation.

(b) Show that N tends to a finite limit N_∞ as $t \to \infty$, and find it.

(c) **[Extension]** Show that your solution is exactly the Gompertz solution of Question 4, with $M = N_\infty$ and $\alpha = a$. What does this tell you about using data to decide between two models?