MATH4120: Mathematical Modelling and Programming

Workshop sheet 4: Solving ODEs

NOT ASSESSED
Week 5 workshops

Chapter 4: choosing a method, first-order ODEs, and second-order linear ODEs with constant coefficients

This sheet is not assessed. It accompanies the Week 5 workshop and gives further practice with the Chapter 4 material, alongside Exercises 4.1–4.5 in the notes. As usual, parts marked **[Extension]** are more challenging: try them once you are comfortable with the rest of the sheet.

Question 1

Group discussion: which method? Do *not* solve these equations. For each one, decide which method from Chapter 4 you would use: separation of variables, integrating factor, Bernoulli substitution, homogeneous substitution $v = y/x$, or something else. Use the checklist in Section 4.6.3.

(i) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = xy^2$ (ii) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + 2y = 6$ (iii) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x-y}{x+y}$

(iv) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + y\tan x = \sin x$ (v) $\dfrac{\mathrm{d}y}{\mathrm{d}x} - y = e^x y^3$ (vi) $x\dfrac{\mathrm{d}y}{\mathrm{d}x} = y + x\cos^2\!\left(\dfrac{y}{x}\right)$

(vii) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = e^{x+y}$ (viii) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = x$ (ix) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x + y^2$

(a) Sort the equations into groups by method.

(b) Some equations fit more than one method. Find them. Which method would you choose, and why?

(c) One equation does not fit any of the first-order methods. Which one, and what does that tell you about ODEs in general? One more is not first order at all: which, and which section of the notes deals with it?

Question 2

Question 3 of Workshop sheet 2 nondimensionalised the temperature of a building whose surroundings warm and cool each day, and obtained

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = -\theta + \cos(\Omega s), \qquad \theta(0) = \theta_0,$$

where $\Omega > 0$ is a constant. It promised that you would be able to solve this in Chapter 4.

(a) Write the equation in the standard form $\theta' + a(s)\,\theta = b(s)$, and find the integrating factor $\mu(s)$.

(b) Multiply through by μ and recognise the left-hand side as a single derivative. Hence write θ in terms of the integral $\int e^s \cos(\Omega s)\,\mathrm{d}s$.

(c) Verify, by differentiating the right-hand side, that

$$\int e^s \cos(\Omega s)\,\mathrm{d}s = \frac{e^s\left(\cos\Omega s + \Omega \sin \Omega s\right)}{1+\Omega^2} + C.$$

(d) Find the solution with $\theta(0) = \theta_0$. What happens to the term involving θ_0 as $s \to \infty$?

(e) **[Extension]** By writing $\cos\Omega s + \Omega\sin\Omega s = R\cos(\Omega s - \phi)$, show that at long times θ oscillates with amplitude $1/\sqrt{1+\Omega^2}$. Evaluate this for the garden shed ($\Omega \approx 0.26$) and the cellar ($\Omega \approx 63$) from sheet 2.

Question 3

A population $N(t)$ with growth rate $r > 0$ and carrying capacity $K > 0$ satisfies

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN\left(1 - \frac{N}{K}\right), \qquad N(0) = N_0 > 0.$$

You will study this model in detail in Chapter 5.

(a) Show that this is a Bernoulli equation, and identify P, Q and n.

(b) Let $v = 1/N$. Use the chain rule to find $\mathrm{d}v/\mathrm{d}t$ in terms of $\mathrm{d}N/\mathrm{d}t$, and show that v satisfies the linear equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} + rv = \frac{r}{K}.$$

(c) Solve for v with an integrating factor, and hence show that

$$N(t) = \frac{KN_0}{N_0 + (K - N_0)e^{-rt}}.$$

(d) What happens to N as $t \to \infty$? Sketch $N(t)$ for one starting value $N_0 < K$ and one $N_0 > K$.

Question 4

Consider

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} + \frac{x}{y}, \qquad y(1) = 2, \qquad x > 0.$$

(a) Show that the equation is homogeneous, i.e. of the form $y' = F(y/x)$, and identify F.

(b) Substitute $y = vx$. Use the product rule to write y' in terms of v and v', and show that $x\,\mathrm{d}v/\mathrm{d}x = 1/v$.

(c) Separate the variables, integrate, and use the initial condition to find $y(x)$ explicitly.

(d) For which values of $x > 0$ does your solution exist? What goes wrong in the ODE at the end of this interval?

Question 5

(a) Write down the auxiliary equation, its roots, and the general solution of each of

(i) $y'' - y' - 6y = 0$, (ii) $y'' + 6y' + 9y = 0$, (iii) $y'' + 2y' + 5y = 0$, (iv) $y'' + 16y = 0$.

Which of the three cases of Section 4.8 does each belong to?

(b) A simple model of one corner of a car treats it as a mass $M = 400$ kg on a spring of stiffness $\kappa = 40{,}000\ \mathrm{N\,m^{-1}}$ with a damper of strength c. After hitting a bump, the vertical displacement $z(t)$ satisfies

$$M\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} + c\frac{\mathrm{d}z}{\mathrm{d}t} + \kappa z = 0.$$

Divide by M and compare with the damped oscillator $z'' + 2kz' + \omega^2 z = 0$ of Section 4.9 to find k and ω in terms of M, c and κ. Evaluate ω.

(c) Classify the damping (under-, critically or over-damped) for dampers with $c = 2400$, 8000 and 16,000 $\mathrm{kg\,s^{-1}}$, and find the roots of the auxiliary equation in each case.

(d) What value of c gives critical damping?

(e) **[Extension]** After a bump, which of the three cars settles back to $z = 0$ most quickly, and which most slowly? Compare the slowest-decaying exponential in each case.

Question 6

(a) The equation $y'' - 3y' + 2y = f(x)$ has auxiliary equation $m^2 - 3m + 2 = 0$, with roots 1 and 2. Without solving anything, write down the trial function you would use for the particular integral when

$$\text{(i) } f = 4x + 2, \quad \text{(ii) } f = 5e^{3x}, \quad \text{(iii) } f = 3e^{x}, \quad \text{(iv) } f = 10\sin x, \quad \text{(v) } f = x^2.$$

(b) Find the general solution of $y'' - 3y' + 2y = 4x + 2$.

(c) *Resonance.* A child on a swing behaves approximately like an undamped oscillator. If it is pushed periodically at exactly its natural frequency ω, its displacement $y(t)$ satisfies

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \omega^2 y = F\cos\omega t,$$

where F and ω are positive constants.

(i) Find the complementary function.

(ii) Explain why the trial function $\phi = A\cos\omega t + B\sin\omega t$ cannot work.

(iii) Use the trial $\phi = t\,(A\cos\omega t + B\sin\omega t)$ to show that the particular integral is $\phi = \dfrac{F}{2\omega}\,t\sin\omega t$.

(iv) Describe what happens to the swing as time goes on.