MATH4120: Mathematical Modelling and Programming

Workshop sheet 4: Solving ODEs

Chapter 4: choosing a method, first-order ODEs, and second-order linear ODEs with constant coefficients

NOT ASSESSED
Week 5 workshops

Model solutions, do not circulate until after the workshops.

This sheet is not assessed. It accompanies the Week 5 workshop and gives further practice with the Chapter 4 material, alongside Exercises 4.1–4.5 in the notes. As usual, parts marked **[Extension]** are more challenging: try them once you are comfortable with the rest of the sheet.

Question 1

Group discussion: which method? Do *not* solve these equations. For each one, decide which method from Chapter 4 you would use: separation of variables, integrating factor, Bernoulli substitution, homogeneous substitution $v = y/x$, or something else. Use the checklist in Section 4.6.3.

(i) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = xy^2$ (ii) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + 2y = 6$ (iii) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x-y}{x+y}$

(iv) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + y\tan x = \sin x$ (v) $\dfrac{\mathrm{d}y}{\mathrm{d}x} - y = e^x y^3$ (vi) $x\dfrac{\mathrm{d}y}{\mathrm{d}x} = y + x\cos^2\!\left(\dfrac{y}{x}\right)$

(vii) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = e^{x+y}$ (viii) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = x$ (ix) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x + y^2$

(a) Sort the equations into groups by method.

(b) Some equations fit more than one method. Find them. Which method would you choose, and why?

(c) One equation does not fit any of the first-order methods. Which one, and what does that tell you about ODEs in general? One more is not first order at all: which, and which section of the notes deals with it?

Solution

(a) *Separable:* (i) $y^{-2}\,\mathrm{d}y = x\,\mathrm{d}x$; (ii) $\mathrm{d}y/(6-2y) = \mathrm{d}x$; (vii) $e^{x+y} = e^x e^y$, so $e^{-y}\,\mathrm{d}y = e^x\,\mathrm{d}x$. *Linear (integrating factor):* (ii) with $a = 2$, $b = 6$; (iv) with $a(x) = \tan x$, $b(x) = \sin x$, giving $\mu = e^{\int \tan x\,\mathrm{d}x} = \sec x$ (for $|x| < \pi/2$). *Bernoulli:* (v) with $P = -1$, $Q = e^x$, $n = 3$; also (i), with $P = 0$, $Q = x$, $n = 2$. *Homogeneous:* (iii) with $F(v) = (1-v)/(1+v)$, after dividing top and bottom by x; (vi) with $F(v) = v + \cos^2 v$, after dividing by x. *Something else:* (viii) and (ix).
(b) (i) is separable and also Bernoulli; (ii) is separable and also linear. In both cases separation is quicker: it needs one integration and no substitution. The categories in Section 4.6.3 overlap, and it is worth checking the simplest method first.
(c) (ix), $y' = x + y^2$, is not separable (the right side is a sum, not a product), not linear (because of y^2), not Bernoulli (the x term has no factor of y), and not homogeneous. In fact it has no solution in terms of elementary functions. Most ODEs cannot be solved by

hand: this is why numerical methods (Chapter 6) matter. (viii) is second order, linear with constant coefficients: it is solved by a complementary function plus a particular integral (Sections 4.7–4.11), giving $y = c_1 \cos x + c_2 \sin x + x$.

Question 2

Question 3 of Workshop sheet 2 nondimensionalised the temperature of a building whose surroundings warm and cool each day, and obtained

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = -\theta + \cos(\Omega s), \qquad \theta(0) = \theta_0,$$

where $\Omega > 0$ is a constant. It promised that you would be able to solve this in Chapter 4.

(a) Write the equation in the standard form $\theta' + a(s)\,\theta = b(s)$, and find the integrating factor $\mu(s)$.

(b) Multiply through by μ and recognise the left-hand side as a single derivative. Hence write θ in terms of the integral $\int e^s \cos(\Omega s)\,\mathrm{d}s$.

(c) Verify, by differentiating the right-hand side, that

$$\int e^s \cos(\Omega s)\,\mathrm{d}s = \frac{e^s(\cos\Omega s + \Omega \sin \Omega s)}{1+\Omega^2} + C.$$

(d) Find the solution with $\theta(0) = \theta_0$. What happens to the term involving θ_0 as $s \to \infty$?

(e) **[Extension]** By writing $\cos\Omega s + \Omega \sin \Omega s = R\cos(\Omega s - \phi)$, show that at long times θ oscillates with amplitude $1/\sqrt{1+\Omega^2}$. Evaluate this for the garden shed ($\Omega \approx 0.26$) and the cellar ($\Omega \approx 63$) from sheet 2.

Solution

(a) $\theta' + \theta = \cos\Omega s$, so $a = 1$, $b = \cos\Omega s$, and $\mu = \exp(\int 1\,\mathrm{d}s) = e^s$.

(b) Multiplying by e^s:

$$e^s\theta' + e^s\theta = e^s\cos\Omega s \quad\Longrightarrow\quad \frac{\mathrm{d}}{\mathrm{d}s}(e^s\theta) = e^s\cos\Omega s,$$

by the product rule. Integrating, $e^s\theta = \int e^s\cos\Omega s\,\mathrm{d}s$, so $\theta = e^{-s}\int e^s\cos\Omega s\,\mathrm{d}s$.

(c) Using the product rule on $e^s(\cos\Omega s + \Omega\sin\Omega s)$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\left[e^s(\cos\Omega s + \Omega\sin\Omega s)\right] &= e^s(\cos\Omega s + \Omega\sin\Omega s) + e^s(-\Omega\sin\Omega s + \Omega^2\cos\Omega s)\\ &= (1+\Omega^2)\,e^s\cos\Omega s.\end{aligned}$$

Dividing by $1+\Omega^2$ gives $e^s\cos\Omega s$. ✓(The formula can be derived by integrating by parts twice.)

(d) From (b) and (c),

$$\theta(s) = \frac{\cos\Omega s + \Omega\sin\Omega s}{1+\Omega^2} + Ce^{-s}.$$

At $s = 0$: $\theta_0 = 1/(1+\Omega^2) + C$, so $C = \theta_0 - 1/(1+\Omega^2)$. The term Ce^{-s} decays to zero as $s \to \infty$, so the starting temperature is forgotten after a few relaxation times, exactly as argued in sheet 2, Question 3(e).

(e) $R\cos(\Omega s-\phi)=R\cos\phi\cos\Omega s+R\sin\phi\sin\Omega s$. Matching coefficients, $R\cos\phi=1$ and $R\sin\phi=\Omega$, so $R=\sqrt{1+\Omega^2}$ and $\tan\phi=\Omega$. The long-time solution is

$$\theta\approx\frac{\sqrt{1+\Omega^2}}{1+\Omega^2}\cos(\Omega s-\phi)=\frac{1}{\sqrt{1+\Omega^2}}\cos(\Omega s-\phi),$$

with amplitude $1/\sqrt{1+\Omega^2}$. Shed: $1/\sqrt{1.068}\approx 0.97$, so it follows nearly the full outdoor swing. Cellar: $1/\sqrt{3970}\approx 0.016$, so a ± 8 °C outdoor swing becomes about ± 0.13 °C. This confirms the limits found in sheet 2 without solving. (The phase lag $\phi=\arctan\Omega$ means the shed is warmest about an hour after the outside is, and the cellar lags by about a quarter of a day.)

Question 3

A population $N(t)$ with growth rate $r>0$ and carrying capacity $K>0$ satisfies

$$\frac{\mathrm{d}N}{\mathrm{d}t}=rN\left(1-\frac{N}{K}\right),\qquad N(0)=N_0>0.$$

You will study this model in detail in Chapter 5.

(a) Show that this is a Bernoulli equation, and identify P, Q and n.

(b) Let $v=1/N$. Use the chain rule to find $\mathrm{d}v/\mathrm{d}t$ in terms of $\mathrm{d}N/\mathrm{d}t$, and show that v satisfies the linear equation

$$\frac{\mathrm{d}v}{\mathrm{d}t}+rv=\frac{r}{K}.$$

(c) Solve for v with an integrating factor, and hence show that

$$N(t)=\frac{KN_0}{N_0+(K-N_0)e^{-rt}}.$$

(d) What happens to N as $t\to\infty$? Sketch $N(t)$ for one starting value $N_0<K$ and one $N_0>K$.

Solution

(a) Expanding, $N'=rN-(r/K)N^2$, i.e.

$$\frac{\mathrm{d}N}{\mathrm{d}t}-rN=-\frac{r}{K}N^2,$$

which is $N'+PN=QN^n$ with $P=-r$, $Q=-r/K$ and $n=2$.

(b) With $v=N^{-1}$, the chain rule gives

$$\frac{\mathrm{d}v}{\mathrm{d}t}=-N^{-2}\frac{\mathrm{d}N}{\mathrm{d}t}.$$

Divide the equation in (a) by N^2: $N^{-2}N'-rN^{-1}=-r/K$. Since $N^{-2}N'=-v'$ and $N^{-1}=v$, this is $-v'-rv=-r/K$, i.e. $v'+rv=r/K$. (This agrees with formula (4.7): $(1-n)P=r$ and $(1-n)Q=r/K$.)

(c) The integrating factor is $\mu=e^{rt}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{rt}v)=\frac{r}{K}e^{rt}\quad\Longrightarrow\quad e^{rt}v=\frac{1}{K}e^{rt}+C\quad\Longrightarrow\quad v=\frac{1}{K}+Ce^{-rt}.$$

At $t = 0$, $v = 1/N_0$, so $C = 1/N_0 - 1/K$. Then

$$N = \frac{1}{v} = \frac{1}{\frac{1}{K} + \left(\frac{1}{N_0} - \frac{1}{K}\right) e^{-rt}} = \frac{KN_0}{N_0 + (K - N_0)e^{-rt}},$$

multiplying top and bottom by KN_0.
(d) $e^{-rt} \to 0$, so $N \to KN_0/N_0 = K$: the population approaches the carrying capacity whatever its (positive) starting value. If $N_0 < K$, N rises along an S-shaped curve towards K; if $N_0 > K$, it decreases towards K. The equation is also separable, and Chapter 5 solves it that way, with partial fractions.

Question 4

Consider

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} + \frac{x}{y}, \qquad y(1) = 2, \qquad x > 0.$$

(a) Show that the equation is homogeneous, i.e. of the form $y' = F(y/x)$, and identify F.

(b) Substitute $y = vx$. Use the product rule to write y' in terms of v and v', and show that $x\,\mathrm{d}v/\mathrm{d}x = 1/v$.

(c) Separate the variables, integrate, and use the initial condition to find $y(x)$ explicitly.

(d) For which values of $x > 0$ does your solution exist? What goes wrong in the ODE at the end of this interval?

Solution

(a) With $v = y/x$, the right-hand side is $v + 1/v$, so $F(v) = v + 1/v$.
(b) $y = vx$ gives, by the product rule, $y' = v + x\,v'$. So $v + xv' = v + 1/v$, i.e. $x\,v' = 1/v$.
(c) Separating,

$$v\,\mathrm{d}v = \frac{\mathrm{d}x}{x} \quad \Longrightarrow \quad \frac{v^2}{2} = \log x + c$$

(no modulus needed, since $x > 0$). At $x = 1$, $v = y/x = 2$, so $2 = 0 + c$ and $c = 2$. Then $v^2 = 2\log x + 4$, and taking the positive root (because $y(1) = 2 > 0$),

$$y = x\sqrt{2\log x + 4}.$$

(d) We need $2\log x + 4 > 0$, i.e. $\log x > -2$, i.e. $x > e^{-2} \approx 0.135$. So the solution exists on the interval (e^{-2}, ∞), not for all $x > 0$. As $x \to e^{-2}$ from above, $y \to 0$, and the term x/y in the ODE blows up, so the slope becomes infinite. This is why a solution is defined on an interval containing the initial point (Section 4.1).

Question 5

(a) Write down the auxiliary equation, its roots, and the general solution of each of

(i) $y'' - y' - 6y = 0$, (ii) $y'' + 6y' + 9y = 0$, (iii) $y'' + 2y' + 5y = 0$, (iv) $y'' + 16y = 0$.

Which of the three cases of Section 4.8 does each belong to?

(b) A simple model of one corner of a car treats it as a mass $M = 400$ kg on a spring of stiffness $\kappa = 40{,}000\ \mathrm{N\,m^{-1}}$ with a damper of strength c. After hitting a bump, the vertical

displacement $z(t)$ satisfies

$$M\frac{\mathrm{d}^2z}{\mathrm{d}t^2}+c\frac{\mathrm{d}z}{\mathrm{d}t}+\kappa z=0.$$

Divide by M and compare with the damped oscillator $z''+2kz'+\omega^2z=0$ of Section 4.9 to find k and ω in terms of M, c and κ. Evaluate ω.

(c) Classify the damping (under-, critically or over-damped) for dampers with $c=2400$, 8000 and 16,000 $\mathrm{kg\,s^{-1}}$, and find the roots of the auxiliary equation in each case.

(d) What value of c gives critical damping?

(e) **[Extension]** After a bump, which of the three cars settles back to $z=0$ most quickly, and which most slowly? Compare the slowest-decaying exponential in each case.

Solution

(a) (i) $m^2-m-6=(m-3)(m+2)=0$, roots $3,-2$ (distinct real): $y=c_1e^{3x}+c_2e^{-2x}$. (ii) $m^2+6m+9=(m+3)^2=0$, root -3 repeated: $y=(c_1+c_2x)e^{-3x}$. (iii) $m^2+2m+5=0$, $m=-1\pm2i$ (complex): $y=e^{-x}(c_1\cos2x+c_2\sin2x)$. (iv) $m^2+16=0$, $m=\pm4i$ (complex, simple harmonic motion): $y=c_1\cos4x+c_2\sin4x$, with period $2\pi/4=\pi/2$.
(b) $z''+(c/M)z'+(\kappa/M)z=0$, so

$$k=\frac{c}{2M},\qquad \omega=\sqrt{\frac{\kappa}{M}}=\sqrt{\frac{40{,}000}{400}}=10\ \mathrm{s^{-1}}.$$

(c) The auxiliary equation is $m^2+2km+100=0$, with roots $m=-k\pm\sqrt{k^2-100}$, and $k=c/800$.

- $c=2400$: $k=3<\omega$, *underdamped*. $m=-3\pm i\sqrt{91}\approx-3\pm9.54i$: the car bounces, with period $2\pi/9.54\approx0.66$ s, inside a decaying envelope e^{-3t}.
- $c=8000$: $k=10=\omega$, *critically damped*. $m=-10$ (repeated): $z=(c_1+c_2t)e^{-10t}$.
- $c=16{,}000$: $k=20>\omega$, *overdamped*. $m=-20\pm\sqrt{300}\approx-2.68$ and -37.3.

(d) Critical damping needs $k=\omega$, i.e. $c=2M\omega=2\times400\times10=8000\ \mathrm{kg\,s^{-1}}$.
(e) The slowest-decaying parts are e^{-3t} (underdamped), e^{-10t} (critical) and $e^{-2.68t}$ (overdamped). So the critically damped car settles fastest, and the overdamped car is slowest of all, even slower than the bouncy one: a very stiff damper makes the car creep back. In practice car suspensions are usually tuned somewhat below critical, accepting a little bounce for a more comfortable ride.

Question 6

(a) The equation $y''-3y'+2y=f(x)$ has auxiliary equation $m^2-3m+2=0$, with roots 1 and 2. Without solving anything, write down the trial function you would use for the particular integral when

$$\text{(i) } f=4x+2,\quad \text{(ii) } f=5e^{3x},\quad \text{(iii) } f=3e^{x},\quad \text{(iv) } f=10\sin x,\quad \text{(v) } f=x^2.$$

(b) Find the general solution of $y''-3y'+2y=4x+2$.

(c) *Resonance.* A child on a swing behaves approximately like an undamped oscillator. If it is pushed periodically at exactly its natural frequency ω, its displacement $y(t)$ satisfies

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2}+\omega^2y=F\cos\omega t,$$

where F and ω are positive constants.

(i) Find the complementary function.

(ii) Explain why the trial function $\phi = A\cos\omega t + B\sin\omega t$ cannot work.

(iii) Use the trial $\phi = t\,(A\cos\omega t + B\sin\omega t)$ to show that the particular integral is $\phi = \dfrac{F}{2\omega}\,t\sin\omega t$.

(iv) Describe what happens to the swing as time goes on.

Solution

(a) (i) $Ax + B$. (ii) Ae^{3x}, since 3 is not a root of the auxiliary equation. (iii) Axe^x: e^x is already in the complementary function (because 1 is a root), so the usual trial Ae^x would give zero when substituted; multiply by x. (iv) $A\cos x + B\sin x$ (both are needed, even though f has only $\sin x$). (v) $Ax^2 + Bx + C$ (all lower powers are needed).

(b) The complementary function is $c_1e^x + c_2e^{2x}$. Try $\phi = Ax + B$, so $\phi' = A$ and $\phi'' = 0$:

$$0 - 3A + 2(Ax + B) = 4x + 2 \quad\Longrightarrow\quad 2Ax + (2B - 3A) = 4x + 2.$$

Matching coefficients of x: $2A = 4$, so $A = 2$. Constants: $2B - 6 = 2$, so $B = 4$. Hence

$$y = c_1e^x + c_2e^{2x} + 2x + 4.$$

(c) (i) The auxiliary equation $m^2 + \omega^2 = 0$ has roots $\pm i\omega$, so the complementary function is $c_1\cos\omega t + c_2\sin\omega t$.
(ii) This trial is exactly the complementary function, so $\phi'' + \omega^2\phi = 0$ for every A and B. It can never equal $F\cos\omega t$. The forcing is at the natural frequency, so we multiply the trial by t (Section 4.10).
(iii) Write $\phi = tg$ with $g = A\cos\omega t + B\sin\omega t$, which satisfies $g'' = -\omega^2 g$. By the product rule, $\phi' = g + tg'$ and $\phi'' = 2g' + tg''$. So

$$\phi'' + \omega^2\phi = 2g' + t(g'' + \omega^2 g) = 2g' = 2\omega(-A\sin\omega t + B\cos\omega t).$$

Setting this equal to $F\cos\omega t$: $-2\omega A = 0$ and $2\omega B = F$, so $A = 0$ and $B = F/(2\omega)$. Hence $\phi = \dfrac{F}{2\omega}\,t\sin\omega t$.
(iv) The general solution is $y = c_1\cos\omega t + c_2\sin\omega t + \dfrac{F}{2\omega}t\sin\omega t$. The last term oscillates with an amplitude $Ft/(2\omega)$ that grows in proportion to t: each push adds a little more energy, so the swing goes higher and higher. This is *resonance*. In reality friction (damping) eventually limits the amplitude.