MATH4120: Mathematical Modelling and Programming

Workshop sheet 3: The Π-theorem in practice

Chapter 3: the dimension matrix and Buckingham's Π-theorem

NOT ASSESSED
Week 4 workshops

Model solutions, do not circulate until after the workshops.

This sheet is not assessed. It accompanies the Week 4 workshop and gives further practice with the Chapter 3 material. Question 6 is a first look at Chapter 4. Parts marked **[Extension]** are more challenging: try them once you are comfortable with the rest of the sheet. Work through as much as you can in the workshop and finish the rest in your own time.

Question 1

Counting dimensionless groups. For each list of governing parameters, write down the dimensions of each parameter, find n (the number of parameters), k (the number with independent dimensions) and $m = n - k$. You do not need to find the groups themselves unless asked.

(a) The period τ of a pendulum, its length l, and g.

(b) Flow past an obstacle: fluid density ρ, dynamic viscosity μ (dimensions $\mathrm{M\,L^{-1}T^{-1}}$), kinematic viscosity $\nu = \mu/\rho$, obstacle size L and flow speed U. Is something wrong with this list?

(c) An explosion: energy released E, air density ρ, time t and fireball radius R. (You will meet this again in Question 4.)

(d) A cooling cup of tea: the time t it takes to cool, the rate constant r in Newton's law of cooling (called k in Workshop sheet 2), the starting temperature difference $\Delta T_0 = T_0 - T_\infty$, and the final temperature difference $\Delta T_1 = T_1 - T_\infty$. Treat temperature as a fourth base dimension, Θ. Find the groups, and compare with Question 5 of Workshop sheet 2.

Solution

(a) $[\tau] = \mathrm{T}$, $[l] = \mathrm{L}$, $[g] = \mathrm{L\,T^{-2}}$. So $n = 3$. Only L and T appear, so $k = 2$ (not 3: mass does not appear in any parameter), and $m = 1$. The group is $\tau\sqrt{g/l}$, as in Section 3.2. A common slip is to write $k = 3$ automatically, which would wrongly give $m = 0$. The number k counts the independent dimensions that actually appear (it is the rank of the dimension matrix).
(b) $[\rho] = \mathrm{M\,L^{-3}}$, $[\mu] = \mathrm{M\,L^{-1}T^{-1}}$, $[\nu] = \mathrm{L^2T^{-1}}$, $[L] = \mathrm{L}$, $[U] = \mathrm{L\,T^{-1}}$. So $n = 5$, $k = 3$ and $m = 2$. The groups can be taken as UL/ν (a Reynolds number) and $\mu/(\rho\nu)$. But $\mu/(\rho\nu) = 1$ always, so it carries no information. The list double-counts: ν is not a separate physical parameter, since it is defined from μ and ρ. Keep either ν, or μ and ρ, but not all three. Then there is a single genuine group, the Reynolds number.
(c) $[E] = \mathrm{M\,L^2T^{-2}}$, $[\rho] = \mathrm{M\,L^{-3}}$, $[t] = \mathrm{T}$, $[R] = \mathrm{L}$. So $n = 4$, $k = 3$ and $m = 1$: this is the Sedov–Taylor example in the notes, with group $\Pi = E^{-1/5}\rho^{1/5}t^{-2/5}R$.
(d) $[t] = \mathrm{T}$, $[r] = \mathrm{T^{-1}}$, $[\Delta T_0] = [\Delta T_1] = \Theta$. So $n = 4$, $k = 2$ (T and Θ) and $m = 2$. The groups are rt and $\Delta T_1/\Delta T_0$, so dimensional analysis gives $rt = F(\Delta T_1/\Delta T_0)$. This matches

the exact result $rt = \ln(\Delta T_0/\Delta T_1)$, and it is why the two cups in sheet 2 collapsed when plotted against the dimensionless time (written $s = kt$ there).

Question 2

Flow through a pipe. Blood flows slowly along an artery of radius R. The volume flow rate Q (in $\mathrm{m}^3\,\mathrm{s}^{-1}$) depends on the pressure drop per unit length, $G = \Delta p/L$, the radius R, and the viscosity μ of the blood.

(a) Write down the dimensions of Q, G, R and μ. Construct the dimension matrix, and show that

$$Q = C\,\frac{GR^4}{\mu}$$

for some dimensionless constant C.

(b) A fatty deposit narrows an artery so that its radius is reduced by 10%. If the pressure gradient is unchanged, by what percentage does the flow fall? By what percentage would the pressure gradient have to rise to restore the original flow?

(c) **[Extension]** Suppose we had also included the blood density ρ in the list. How many dimensionless groups would there be now? Find a second group that involves ρ, and explain why it is reasonable to leave ρ out when the flow is slow.

Solution

(a) $[Q] = \mathrm{L}^3\mathrm{T}^{-1}$. Pressure is force per area, $\mathrm{M}\,\mathrm{L}^{-1}\mathrm{T}^{-2}$, so $[G] = \mathrm{M}\,\mathrm{L}^{-2}\mathrm{T}^{-2}$. $[R] = \mathrm{L}$ and $[\mu] = \mathrm{M}\,\mathrm{L}^{-1}\mathrm{T}^{-1}$. So $n = 4$, $k = 3$ and $m = 1$. With $\Pi = Q^{k_1}G^{k_2}R^{k_3}\mu^{k_4}$:

	Q (k_1)	G (k_2)	R (k_3)	μ (k_4)
M	0	1	0	1
L	3	−2	1	−1
T	−1	−2	0	−1

$$\begin{aligned} \mathrm{M}: &\ k_2 + k_4 = 0, \\ \mathrm{L}: &\ 3k_1 - 2k_2 + k_3 - k_4 = 0, \\ \mathrm{T}: &\ -k_1 - 2k_2 - k_4 = 0. \end{aligned}$$

Put $k_1 = 1$. M gives $k_4 = -k_2$, so T gives $-1 - k_2 = 0$, i.e. $k_2 = -1$ and $k_4 = 1$. Then L gives $3 + 2 + k_3 - 1 = 0$, so $k_3 = -4$. Hence $\Pi = Q\mu/(GR^4)$, and $\Pi = C$ gives $Q = CGR^4/\mu$. (Solving the fluid equations gives $C = \pi/8$; this is Poiseuille's law.)

(b) $Q \propto R^4$, so the flow is multiplied by $0.9^4 \approx 0.656$: a fall of about 34% from only a 10% narrowing. To restore the flow, G must be multiplied by $1/0.656 \approx 1.52$, a rise of about 52%. This strong dependence on radius is why narrowed arteries are so serious.

(c) Now $n = 5$ and $k = 3$, so $m = 2$. A second group is

$$\Pi_2 = \frac{\rho Q}{\mu R},$$

since $[\rho Q] = \mathrm{M}\,\mathrm{L}^{-3} \cdot \mathrm{L}^3\mathrm{T}^{-1} = \mathrm{M}\,\mathrm{T}^{-1} = [\mu R]$. Writing $Q \sim UR^2$ for a typical speed U, $\Pi_2 \sim \rho UR/\mu$, which is a Reynolds number. It compares inertia with viscous forces. For slow flow in a narrow tube it is small, so inertia (and hence ρ) has a negligible effect, and the relation reduces to the one-group result of (a).

Question 3

Water waves. The speed c of waves on water depends on which physical effects are important.

(a) For long waves on deep water, gravity is the dominant restoring force, and c depends only on the wavelength λ and g. Show that $c = C_1\sqrt{g\lambda}$. Do long waves travel faster or slower than short ones?

(b) For very short ripples, surface tension σ (force per unit length) dominates, and c depends on λ, σ and the density ρ. Show that $c = C_2\sqrt{\sigma/(\rho\lambda)}$. Do long ripples travel faster or slower than short ones?

(c) In general, the period τ of a water wave depends on λ, the water depth h, g, ρ and σ. This is the water-waves exercise in Chapter 3 of the notes. How many dimensionless groups are there? Construct the dimension matrix and find a set of groups.

(d) **[Extension]** Using $c = \lambda/\tau$, explain how your answers to (a) and (b) arise as special cases of (c). Use the group involving σ to estimate the wavelength at which gravity and surface tension are equally important, for water with $\sigma = 0.073\ \mathrm{N\,m^{-1}}$ and $\rho = 1000\ \mathrm{kg\,m^{-3}}$.

Solution

(a) $[c] = \mathrm{L\,T^{-1}}$, $[\lambda] = \mathrm{L}$, $[g] = \mathrm{L\,T^{-2}}$: $n = 3$, $k = 2$, $m = 1$. The group is $c/\sqrt{g\lambda}$, so $c = C_1\sqrt{g\lambda}$. Long waves travel faster: this is why ocean swell from distant storms arrives with the longest waves first. (Full theory gives $C_1 = 1/\sqrt{2\pi}$.)
(b) $[\sigma] = \mathrm{M\,L\,T^{-2}/L} = \mathrm{M\,T^{-2}}$ and $[\rho] = \mathrm{M\,L^{-3}}$: $n = 4$, $k = 3$, $m = 1$. Since $[\sigma/(\rho\lambda)] = \mathrm{M\,T^{-2}/(M\,L^{-3}\cdot L)} = \mathrm{L^2T^{-2}}$, the group is $c/\sqrt{\sigma/(\rho\lambda)}$, so $c = C_2\sqrt{\sigma/(\rho\lambda)}$. Here shorter ripples travel faster, the opposite of (a). (Full theory gives $C_2 = \sqrt{2\pi}$.)
(c) $[\tau] = \mathrm{T}$, $[\lambda] = [h] = \mathrm{L}$, $[g] = \mathrm{L\,T^{-2}}$, $[\rho] = \mathrm{M\,L^{-3}}$, $[\sigma] = \mathrm{M\,T^{-2}}$. So $n = 6$, $k = 3$, and there are $m = 3$ groups. One convenient set (others are equally valid) is

$$\Pi_1 = \tau\sqrt{\frac{g}{\lambda}}, \qquad \Pi_2 = \frac{h}{\lambda}, \qquad \Pi_3 = \frac{\sigma}{\rho g\lambda^2},$$

found by putting $(k_\tau, k_h, k_\sigma) = (1,0,0)$, $(0,1,0)$ and $(0,0,1)$ in turn. The Π-theorem gives $\tau\sqrt{g/\lambda} = F\big(h/\lambda,\ \sigma/(\rho g\lambda^2)\big)$. Note that ρ can only appear alongside σ, since these are the only two parameters that involve mass.
(d) On deep water, h/λ is large and the depth stops mattering. If also $\Pi_3 \ll 1$ (long waves, surface tension negligible), F is a constant, so $\tau \propto \sqrt{\lambda/g}$ and $c = \lambda/\tau \propto \sqrt{g\lambda}$, which is (a). If instead $\Pi_3 \gg 1$ (short ripples), gravity should drop out. The only way g can cancel from $\tau\sqrt{g/\lambda} = F(\Pi_3)$ is if $F \propto \Pi_3^{-1/2}$, giving $\tau \propto \sqrt{\rho\lambda^3/\sigma}$ and $c \propto \sqrt{\sigma/(\rho\lambda)}$, which is (b).
The two effects are comparable when $\Pi_3 \approx 1$, i.e. $\lambda \approx \sqrt{\sigma/(\rho g)} \approx 3$ mm. Dimensional analysis only fixes this up to a numerical factor: the full theory puts the crossover at $2\pi\sqrt{\sigma/(\rho g)} \approx 1.7$ cm. Either way, waves much shorter than a centimetre or so are ripples controlled by surface tension, and longer waves are controlled by gravity.

Question 4

Taylor's estimate of the Trinity yield. The notes derive the Sedov–Taylor law for the radius of an explosion fireball,

$$R = \Pi\left(\frac{Et^2}{\rho}\right)^{1/5},$$

where Π is a dimensionless constant. In 1950 G. I. Taylor measured the fireball radius from declassified photographs of the first atomic test. Some of his measurements are:

t (ms)	0.38	1.08	4.07	15.0	25.0	62.0
R (m)	25.4	38.9	64.3	106.5	130.0	185.0

(a) If the Sedov–Taylor law holds, what should be true of the ratio $R/t^{2/5}$? Compute it for each measurement (with t in seconds) and comment.

(b) Taylor's full calculation showed that Π is close to 1. Taking $\Pi = 1$ and $\rho = 1.2\ \mathrm{kg\,m^{-3}}$, use the measurement at $t = 25$ ms to estimate the energy E released. Express your answer in kilotons of TNT, where 1 kiloton is 4.2×10^{12} J.

(c) Suppose the radius had been measured 5% too large. How large would the error in E be? What does this tell you about how accurately the photographs needed to be measured?

(d) **[Extension]** The derivation assumed that the pressure of the surrounding air, $p_0 \approx 10^5$ Pa, is negligible compared with the pressure inside the fireball. If p_0 is added to the list of governing parameters, how many groups are there? Show that $p_0 R^3/E$ is one of them, and use it to estimate the radius beyond which the Sedov–Taylor law should stop working. Is Taylor's data safely inside the range of validity?

Solution

(a) The law says $R = \Pi (E/\rho)^{1/5} t^{2/5}$, so $R/t^{2/5} = \Pi (E/\rho)^{1/5}$ should be the same for every measurement. The values (in $\mathrm{m\,s^{-2/5}}$) are

$$593, \quad 598, \quad 581, \quad 571, \quad 569, \quad 563.$$

These agree to within about 3% either side of their mean, over a time range of more than a factor of 100: strong evidence for $R \propto t^{2/5}$. (There is a slight downward drift at late times; see (d).) Equivalently, the slope of $\ln R$ against $\ln t$ between neighbouring points is between 0.38 and 0.41, close to 2/5.

(b) Rearranging with $\Pi = 1$, $E = \rho R^5/t^2$. At $t = 0.025$ s and $R = 130$ m,

$$E = \frac{1.2 \times 130^5}{0.025^2} = \frac{1.2 \times 3.71 \times 10^{10}}{6.25 \times 10^{-4}} \approx 7.1 \times 10^{13}\ \mathrm{J} \approx 17 \text{ kilotons}.$$

Using the other measurements gives between about 16 and 21 kilotons. The accepted yield of the test is roughly 20–25 kilotons, so a few lines of dimensional analysis plus some photographs got remarkably close to a closely guarded secret.

(c) $E \propto R^5$, so a 5% overestimate of R multiplies E by $1.05^5 \approx 1.28$, an error of about 28%. Small errors in R are amplified fivefold (roughly), so the radius had to be measured carefully from the photographs. By contrast, $E \propto t^{-2}$, so errors in t are only doubled.

(d) Now $n = 5$ and $k = 3$, so $m = 2$. Since $[p_0] = \mathrm{M\,L^{-1}T^{-2}}$, $[p_0 R^3] = \mathrm{M\,L^2T^{-2}} = [E]$, so $p_0 R^3/E$ is dimensionless. The law should hold while this group is small, i.e. while the fireball's energy per unit volume, roughly E/R^3, is much larger than p_0. It breaks down when $p_0 R^3/E \approx 1$, at

$$R \approx \left(\frac{E}{p_0}\right)^{1/3} \approx \left(\frac{7 \times 10^{13}}{10^5}\right)^{1/3} \approx 900\ \mathrm{m}.$$

Taylor's largest radius is 185 m, well inside this limit, although the slight drift in (a) may be the first sign of the air pressure starting to matter.

Question 5

Ship models. Engineers test scale models of ships in long water tanks. The main resistance on a ship comes from the waves it makes, and this wave drag D depends on the water density ρ, the ship's speed V, its length L, and g.

(a) Show that dimensional analysis gives

$$\frac{D}{\rho V^2 L^2} = f(Fr), \qquad Fr = \frac{V}{\sqrt{gL}},$$

where f is an unknown function and Fr is called the *Froude number*.

(b) A ship is designed to sail at $10\ \mathrm{m\,s^{-1}}$. A 1:25 scale model is tested in the same water. At what speed must the model be towed so that it makes the same pattern of waves as the full-size ship?

Solution

(a) $[D] = \mathrm{M\,L\,T^{-2}}$, $[\rho] = \mathrm{M\,L^{-3}}$, $[V] = \mathrm{L\,T^{-1}}$, $[L] = \mathrm{L}$, $[g] = \mathrm{L\,T^{-2}}$. So $n = 5$, $k = 3$ and $m = 2$. Exactly as for the drag on a sphere in Section 3.5, the first group is $\Pi_1 = D/(\rho V^2 L^2)$. For the second, choose a group without D. Then ρ is the only remaining parameter involving mass, so it cannot appear either, and we combine V, L and g: $V^2/(gL)$ is dimensionless, or equivalently $V/\sqrt{gL}$. The Π-theorem then gives $\Pi_1 = f(Fr)$.
(b) The model and ship behave alike if they have the same Froude number:

$$\frac{V_\mathrm{m}}{\sqrt{gL_\mathrm{m}}} = \frac{V_\mathrm{s}}{\sqrt{gL_\mathrm{s}}} \quad\Longrightarrow\quad V_\mathrm{m} = V_\mathrm{s}\sqrt{\frac{L_\mathrm{m}}{L_\mathrm{s}}} = 10 \times \sqrt{\frac{1}{25}} = 2\ \mathrm{m\,s^{-1}}.$$

The model is towed much more slowly than the ship sails, because a small ship makes small waves, and small waves travel slowly (compare Question 3(a)).

Question 6

Looking ahead to Chapter 4.

(a) State the order of each ODE, and whether it is linear or nonlinear:

(i) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x^2 y$, (ii) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\dfrac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0$, (iii) $\dfrac{\mathrm{d}V}{\mathrm{d}S} = 1 - V^2$, (iv) $\dfrac{\mathrm{d}^2\theta}{\mathrm{d}s^2} + \sin\theta = 0$.

(b) Equation (iii) is the dimensionless quadratic-drag equation from Question 4 of Workshop sheet 2. Solve it with $V(0) = 0$ by separating variables and using partial fractions, and show that $V = \tanh S$.

(c) Find the value of S at which V reaches 99% of the terminal speed. For linear drag, the corresponding value was $\ln 100 \approx 4.6$. For the skydiver of sheet 2 (time scale $T \approx 5.7$ s), how many seconds does it take to reach 99% of terminal speed?

Solution

(a) (i) First order, linear. (ii) Second order, linear. (iii) First order, nonlinear (because of the V^2). (iv) Second order, nonlinear (because of $\sin\theta$); this is the undamped pendulum.

(b) Separating, $\displaystyle\int \frac{\mathrm{d}V}{1-V^2} = \int \mathrm{d}S$. Since $1-V^2 = (1-V)(1+V)$, partial fractions give

$$\frac{1}{1-V^2} = \frac{1}{2}\left(\frac{1}{1-V} + \frac{1}{1+V}\right),$$

so

$$\frac{1}{2}\ln\left(\frac{1+V}{1-V}\right) = S + c,$$

and $V(0) = 0$ gives $c = 0$. Then $(1+V)/(1-V) = e^{2S}$, which rearranges to

$$V = \frac{e^{2S}-1}{e^{2S}+1} = \frac{e^{S}-e^{-S}}{e^{S}+e^{-S}} = \tanh S.$$

(c) $V = 0.99$ gives $(1+V)/(1-V) = 1.99/0.01 = 199$, so $S = \frac{1}{2}\ln 199 \approx 2.65$. This is much sooner than 4.6 for linear drag, as predicted in sheet 2. For the skydiver, $t = 2.65 \times 5.7 \approx 15$ s.