MATH4120: Mathematical Modelling and Programming

Workshop sheet 2: Scaling and solution collapse

Chapter 2: scales, nondimensionalisation and solution collapse

NOT ASSESSED
Week 3 workshops

This sheet is not assessed. It accompanies the Week 3 workshop and gives further practice with the Chapter 2 material, alongside Exercises 2.1 and 2.2 in the notes. Question 1 is designed for discussion in small groups, so start there. Question 6 uses the dimension matrix from Chapter 3: attempt it if it has been covered in lectures, otherwise come back to it next week. Parts marked **[Extension]** are more challenging: try them once you are comfortable with the rest of the sheet. Work through as much as you can in the workshop and finish the rest in your own time.

Question 1

Group discussion: one pendulum, three time scales. A pendulum of mass m on a light rod of length L swings in a fluid that resists its motion. As in Exercise 2.1 of the notes, its angle $\theta(t)$ from the vertical satisfies

$$mL^2\ddot{\theta} + bL^2\dot{\theta} + mgL\sin\theta = 0,$$

where $b > 0$ measures the strength of the damping.

(a) Before doing any mathematics, sketch $\theta(t)$ for the pendulum released from rest at $\theta = 30^\circ$ when it swings in air, in water, and in honey.

(b) Exercise 2.1 rescales time by $\sqrt{L/g}$ and obtains

$$\theta'' + \gamma\,\theta' + \sin\theta = 0, \qquad \gamma = \frac{b}{m}\sqrt{\frac{L}{g}},$$

where primes denote derivatives with respect to the new dimensionless time. Check that m/b has dimensions of time, and explain why γ is the ratio of two time scales. Which of your three sketches has the smallest γ?

(c) Now rescale time by m/b instead. Show that this gives

$$\theta'' + \theta' + \frac{1}{\gamma^2}\sin\theta = 0.$$

Is this a different model? How many dimensionless groups does it contain?

(d) **[Extension]** A good choice of scales makes the terms that matter of size about one, and makes any small terms visibly small. Which of the two scalings is the better choice for the pendulum in air ($\gamma \ll 1$)? In honey ($\gamma \gg 1$), argue that *neither* works, and find a third time scale that does. (Hint: in honey, which two forces balance?)

(e) **[Extension]** Summarise as a group: what is a free choice when we nondimensionalise, and what is not?

Question 2

The draining tank, collapsed. Section 1.5 of the notes models a cylindrical tank of cross-sectional area A draining through a hole of area a:

$$A\frac{\mathrm{d}h}{\mathrm{d}t} = -a\sqrt{2gh}, \qquad h(0) = h_0.$$

(a) Let $H = h/h_0$ and $s = t/T$, where the time scale T is still to be chosen. Show that $\dfrac{\mathrm{d}H}{\mathrm{d}s} = -\dfrac{aT\sqrt{2g}}{A\sqrt{h_0}}\sqrt{H}$, and choose T so that the coefficient is 1. Check that your T has dimensions of time.

(b) Solve $\mathrm{d}H/\mathrm{d}s = -\sqrt{H}$ with $H(0) = 1$. Show that every such tank empties at $s = 2$, and check that this agrees with the drain time (1.4) in the notes.

(c) Tank 1 has $A = 0.5$ m^2, $a = 2$ cm^2 and $h_0 = 1$ m. Tank 2 has $A = 2$ m^2, $a = 4$ cm^2 and $h_0 = 0.25$ m. Using only your answers to (a) and (b), decide which tank empties first.

(d) **[Extension]** Question 4 of Workshop sheet 1 added a constant inflow Q:

$$A\frac{\mathrm{d}h}{\mathrm{d}t} = Q - a\sqrt{2gh}, \qquad h(0) = h_0,$$

with equilibrium depth $h^* = (Q/a)^2/(2g)$. Scale h by h^* and time by $T = Ah^*/Q$, and show that

$$\frac{\mathrm{d}H}{\mathrm{d}s} = 1 - \sqrt{H}, \qquad H(0) = \frac{h_0}{h^*}.$$

Of the five parameters A, a, g, Q, h_0, how many dimensionless combinations survive? Interpret the time scale Ah^*/Q.

Question 3

Cooling with a daily cycle. A building has internal temperature $T(t)$. The outside temperature varies through the day as $T_{\text{out}}(t) = \bar{T} + A\cos\omega t$, where $\omega = 2\pi/(24\text{ h})$. Newton's law of cooling gives

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -k(T - \bar{T} - A\cos\omega t), \qquad T(0) = T_0.$$

(a) Let $\theta = (T - \bar{T})/A$ and $s = kt$. Show that

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = -\theta + \cos(\Omega s), \qquad \theta(0) = \theta_0,$$

and find Ω and θ_0 in terms of the original parameters.

(b) Explain why Ω compares two time scales. Which ones?

(c) Without solving the ODE, describe the behaviour of $\theta(s)$ when $\Omega \ll 1$ and when $\Omega \gg 1$. (Hint: when $\Omega \gg 1$, how much does $\cos(\Omega s)$ push θ in one direction before it reverses?)

(d) A garden shed has $k \approx 1$ h^{-1}, and a deep cellar has $k \approx 1/(10\text{ days})$. Find Ω for each. On a day when the outside temperature swings by ± 8 °C, roughly how much does each swing?

(e) Two identical buildings start at different temperatures. By considering the difference $w = \theta_1 - \theta_2$ between their dimensionless temperatures, show that they eventually behave identically. So does θ_0 matter at long times?

Question 4

Falling with quadratic drag. Question 7 of Workshop sheet 1 suggested replacing linear drag with quadratic drag for larger objects:

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - cv^2, \qquad v(0) = 0.$$

(a) Find the dimensions of c, and show that the terminal velocity is $v_T = \sqrt{mg/c}$.

(b) Let $V = v/v_T$ and $S = t/T$. Show that choosing $T = v_T/g$ gives

$$\frac{\mathrm{d}V}{\mathrm{d}S} = 1 - V^2, \qquad V(0) = 0,$$

an equation with no parameters at all. Interpret T physically.

(c) A skydiver has $m = 80$ kg and $c = 0.25\ \mathrm{kg\,m^{-1}}$. Evaluate v_T and T. If the mass were doubled with c unchanged, by what factors would v_T and T change?

(d) For linear drag ($m\,\mathrm{d}v/\mathrm{d}t = mg - bv$, sheet 1 Question 2), the same procedure gives $\mathrm{d}V/\mathrm{d}S = 1 - V$. Without solving either equation, show that in dimensionless terms the quadratic-drag faller always approaches terminal speed at least as fast as the linear-drag one. (Hint: write $1 - V^2 = (1 - V)(1 + V)$.) Sketch both curves $V(S)$ on the same axes.

Question 5

Collapsing data. Two cups of tea cool in different rooms. Cup A starts at 90 °C in a room at 20 °C; cup B starts at 70 °C in a cooler room at 15 °C. The measured temperatures are:

Cup A: t (min)	0	2	4	6	8	10	15	20
T (°C)	90.0	77.5	67.5	59.0	51.0	46.0	35.5	29.0
Cup B: t (min)	0	4	8	12	16	20	30	40
T (°C)	70.0	60.5	52.0	45.5	40.0	35.0	27.5	22.5

(a) Plot T against t for both cups on the same axes.

(b) For each reading, compute the dimensionless temperature $\theta = (T - T_\infty)/(T_0 - T_\infty)$.

(c) Assuming Newton's law of cooling, estimate the cooling rate k for each cup, either from the slope of $\ln\theta$ against t or from the time taken for θ to halve.

(d) Using your values of k, plot θ against $s = kt$ for both cups on the same axes, and add the curve e^{-s}. What do you see?

Question 6

A first dimension matrix. (Attempt this if Section 3.2 has been covered in lectures.) A mass m on a spring of stiffness κ (force per unit extension) oscillates with amplitude A. We want to know how its period τ depends on m, κ and A.

(a) Write down the dimensions of τ, m, κ and A.

(b) Construct the dimension matrix, find the number of dimensionless groups, and solve for the exponents to show that $\tau = C\sqrt{m/\kappa}$ for some dimensionless constant C.

(c) Explain, by looking at the dimension matrix, why the amplitude A cannot appear in the answer.

(d) **[Extension]** For a pendulum, Section 3.1 of the notes set aside the swing amplitude α. Including it, show that dimensional analysis gives $\tau = \sqrt{l/g}\,F(\alpha)$ for some unknown function F. Why is the amplitude allowed to appear here but not for the spring?