MATH4120: Mathematical Modelling and Programming

Workshop sheet 2: Scaling and solution collapse

Chapter 2: scales, nondimensionalisation and solution collapse

NOT ASSESSED
Week 3 workshops

Model solutions, do not circulate until after the workshops.

This sheet is not assessed. It accompanies the Week 3 workshop and gives further practice with the Chapter 2 material, alongside Exercises 2.1 and 2.2 in the notes. Question 1 is designed for discussion in small groups, so start there. Question 6 uses the dimension matrix from Chapter 3: attempt it if it has been covered in lectures, otherwise come back to it next week. Parts marked **[Extension]** are more challenging: try them once you are comfortable with the rest of the sheet. Work through as much as you can in the workshop and finish the rest in your own time.

Question 1

Group discussion: one pendulum, three time scales. A pendulum of mass m on a light rod of length L swings in a fluid that resists its motion. As in Exercise 2.1 of the notes, its angle $\theta(t)$ from the vertical satisfies

$$mL^2\ddot{\theta} + bL^2\dot{\theta} + mgL\sin\theta = 0,$$

where $b > 0$ measures the strength of the damping.

(a) Before doing any mathematics, sketch $\theta(t)$ for the pendulum released from rest at $\theta = 30°$ when it swings in air, in water, and in honey.

(b) Exercise 2.1 rescales time by $\sqrt{L/g}$ and obtains

$$\theta'' + \gamma\,\theta' + \sin\theta = 0, \qquad \gamma = \frac{b}{m}\sqrt{\frac{L}{g}},$$

where primes denote derivatives with respect to the new dimensionless time. Check that m/b has dimensions of time, and explain why γ is the ratio of two time scales. Which of your three sketches has the smallest γ?

(c) Now rescale time by m/b instead. Show that this gives

$$\theta'' + \theta' + \frac{1}{\gamma^2}\sin\theta = 0.$$

Is this a different model? How many dimensionless groups does it contain?

(d) **[Extension]** A good choice of scales makes the terms that matter of size about one, and makes any small terms visibly small. Which of the two scalings is the better choice for the pendulum in air ($\gamma \ll 1$)? In honey ($\gamma \gg 1$), argue that *neither* works, and find a third time scale that does. (Hint: in honey, which two forces balance?)

(e) **[Extension]** Summarise as a group: what is a free choice when we nondimensionalise, and what is not?

Solution

(a) In air: many swings, with an amplitude that dies away slowly. In water: a few swings that die away quickly. In honey: no swinging at all; the pendulum creeps slowly back to $\theta = 0$ without overshooting.

(b) The b-term is a torque, so $[b] = \mathrm{M\,L^2T^{-2}/(L^2T^{-1})} = \mathrm{M\,T^{-1}}$ and $[m/b] = \mathrm{T}$. Writing

$$\gamma = \frac{\sqrt{L/g}}{m/b} = \frac{\text{swinging time scale}}{\text{damping time scale}},$$

γ compares how long one swing takes with how long the damping takes to remove the motion. Air has the smallest γ: the damping time is many swing times, so we see many swings.

(c) Dividing the ODE by mL^2 gives $\ddot\theta + (b/m)\dot\theta + (g/L)\sin\theta = 0$. With $t = (m/b)s$, each time derivative brings a factor b/m:

$$\frac{b^2}{m^2}\theta'' + \frac{b^2}{m^2}\theta' + \frac{g}{L}\sin\theta = 0 \quad\Longrightarrow\quad \theta'' + \theta' + \frac{gm^2}{Lb^2}\sin\theta = 0,$$

and $gm^2/(Lb^2) = 1/\gamma^2$. It is the same model in different units of time, and it contains the same single group γ (just in the form $1/\gamma^2$).

(d) *Air,* $\gamma \ll 1$. The first scaling is right: θ'' and $\sin\theta$ are of size one and balance (the swinging), while $\gamma\theta'$ is visibly a small correction that slowly drains energy over many swings. In the second scaling the coefficient $1/\gamma^2$ is huge, a sign that the time unit m/b is far too long to resolve individual swings.

Honey, $\gamma \gg 1$. The first scaling has a huge coefficient γ. The second makes the restoring term $\gamma^{-2}\sin\theta$ look negligible, yet that term is what returns the pendulum to vertical. The balance is between damping and gravity, with inertia negligible: $(b/m)\dot\theta \approx -(g/L)\sin\theta$. This gives the time scale

$$t_{\text{creep}} = \frac{bL}{mg} = \gamma\sqrt{\frac{L}{g}},$$

and rescaling time by it gives $\gamma^{-2}\theta'' + \theta' + \sin\theta = 0$, in which inertia is the visibly small term. All three time scales are $\sqrt{L/g}$ multiplied by γ^{-1}, 1 and γ. In honey the speed adjusts on the very short scale m/b, and the pendulum then creeps back on the very long scale $bL/(mg)$.

(e) The choice of scales is free: any choice gives a correct dimensionless equation, and a good choice makes the important balance obvious. What is *not* free is the number of dimensionless groups: every scaling contains γ and nothing else, because the physics depends only on γ.

Question 2

The draining tank, collapsed. Section 1.5 of the notes models a cylindrical tank of cross-sectional area A draining through a hole of area a:

$$A\frac{\mathrm{d}h}{\mathrm{d}t} = -a\sqrt{2gh}, \qquad h(0) = h_0.$$

(a) Let $H = h/h_0$ and $s = t/T$, where the time scale T is still to be chosen. Show that $\dfrac{\mathrm{d}H}{\mathrm{d}s} = -\dfrac{aT\sqrt{2g}}{A\sqrt{h_0}}\sqrt{H}$, and choose T so that the coefficient is 1. Check that your T has dimensions of time.

(b) Solve $\mathrm{d}H/\mathrm{d}s = -\sqrt{H}$ with $H(0) = 1$. Show that every such tank empties at $s = 2$, and check that this agrees with the drain time (1.4) in the notes.

(c) Tank 1 has $A = 0.5$ m^2, $a = 2$ cm^2 and $h_0 = 1$ m. Tank 2 has $A = 2$ m^2, $a = 4$ cm^2 and $h_0 = 0.25$ m. Using only your answers to (a) and (b), decide which tank empties first.

(d) **[Extension]** Question 4 of Workshop sheet 1 added a constant inflow Q:

$$A\frac{\mathrm{d}h}{\mathrm{d}t} = Q - a\sqrt{2gh}, \qquad h(0) = h_0,$$

with equilibrium depth $h^* = (Q/a)^2/(2g)$. Scale h by h^* and time by $T = Ah^*/Q$, and show that

$$\frac{\mathrm{d}H}{\mathrm{d}s} = 1 - \sqrt{H}, \qquad H(0) = \frac{h_0}{h^*}.$$

Of the five parameters A, a, g, Q, h_0, how many dimensionless combinations survive? Interpret the time scale Ah^*/Q.

Solution

(a) With $h = h_0 H$ and $t = Ts$, $\mathrm{d}h/\mathrm{d}t = (h_0/T)\,\mathrm{d}H/\mathrm{d}s$ and $\sqrt{h} = \sqrt{h_0}\sqrt{H}$, so

$$\frac{Ah_0}{T}\frac{\mathrm{d}H}{\mathrm{d}s} = -a\sqrt{2gh_0}\,\sqrt{H} \quad\Longrightarrow\quad \frac{\mathrm{d}H}{\mathrm{d}s} = -\frac{aT\sqrt{2g}}{A\sqrt{h_0}}\sqrt{H}.$$

Setting the coefficient to 1 gives

$$T = \frac{A}{a}\sqrt{\frac{h_0}{2g}}.$$

A/a is dimensionless and $[h_0/g] = \mathrm{L}/(\mathrm{L\,T^{-2}}) = \mathrm{T}^2$, so $[\sqrt{h_0/(2g)}] = \mathrm{T}$. Hence the time scale T has dimensions of time. ✓

(b) Separating, $\int H^{-1/2}\,\mathrm{d}H = -\int \mathrm{d}s$, so $2\sqrt{H} = 2 - s$ and

$$H(s) = \left(1 - \frac{s}{2}\right)^2, \qquad 0 \le s \le 2.$$

This reaches zero at $s = 2$ whatever the tank, so $t_{\mathrm{drain}} = 2T = \dfrac{2A}{a}\sqrt{\dfrac{h_0}{2g}}$, which is (1.4). The notes' solution (1.3), $h/h_0 = (1 - t/t_{\mathrm{drain}})^2$, is this same universal curve.

(c) Tank 1: $A/a = 2500$ and $\sqrt{h_0/(2g)} \approx 0.226$ s, so $T \approx 564$ s. Tank 2: $A/a = 5000$ and $\sqrt{h_0/(2g)} \approx 0.113$ s, so again $T \approx 564$ s. The two tanks look very different, but they have the same time scale, so they empty at the same moment, $2T \approx 1130$ s (about 19 minutes). Once the problem is nondimensionalised, T is the only thing that distinguishes one tank from another.

(d) With $h = h^* H$ and $t = Ts$,

$$\frac{Ah^*}{T}\frac{\mathrm{d}H}{\mathrm{d}s} = Q - a\sqrt{2gh^*}\sqrt{H} = Q(1 - \sqrt{H}),$$

using $a\sqrt{2gh^*} = Q$. Choosing $T = Ah^*/Q$ gives $\mathrm{d}H/\mathrm{d}s = 1 - \sqrt{H}$, with $H(0) = h_0/h^*$. Only one dimensionless combination survives, h_0/h^*, and it enters only through the initial condition. The time scale Ah^*/Q is the time the inflow alone would take to fill the empty tank to its equilibrium depth. (This confirms Question 4(e) of sheet 1: A sets the speed of approach to equilibrium, not the equilibrium itself.)

Question 3

Cooling with a daily cycle. A building has internal temperature $T(t)$. The outside temper-

ature varies through the day as $T_{\text{out}}(t) = \bar{T} + A\cos\omega t$, where $\omega = 2\pi/(24\text{ h})$. Newton's law of cooling gives

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -k(T - \bar{T} - A\cos\omega t), \qquad T(0) = T_0.$$

(a) Let $\theta = (T - \bar{T})/A$ and $s = kt$. Show that

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = -\theta + \cos(\Omega s), \qquad \theta(0) = \theta_0,$$

and find Ω and θ_0 in terms of the original parameters.

(b) Explain why Ω compares two time scales. Which ones?

(c) Without solving the ODE, describe the behaviour of $\theta(s)$ when $\Omega \ll 1$ and when $\Omega \gg 1$. (Hint: when $\Omega \gg 1$, how much does $\cos(\Omega s)$ push θ in one direction before it reverses?)

(d) A garden shed has $k \approx 1\text{ h}^{-1}$, and a deep cellar has $k \approx 1/(10\text{ days})$. Find Ω for each. On a day when the outside temperature swings by $\pm 8\,^\circ$C, roughly how much does each swing?

(e) Two identical buildings start at different temperatures. By considering the difference $w = \theta_1 - \theta_2$ between their dimensionless temperatures, show that they eventually behave identically. So does θ_0 matter at long times?

Solution

(a) With $T = \bar{T} + A\theta$ and $t = s/k$, $\mathrm{d}T/\mathrm{d}t = Ak\,\mathrm{d}\theta/\mathrm{d}s$, and the right-hand side is $-k(A\theta - A\cos\omega t)$. Dividing by Ak and writing $\omega t = (\omega/k)s$,

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = -\theta + \cos(\Omega s), \qquad \Omega = \frac{\omega}{k}, \qquad \theta_0 = \frac{T_0 - \bar{T}}{A}.$$

Five parameters (k, $\bar{T}$, A, ω, T_0) have become two dimensionless groups.
(b) $\Omega = (1/k)/(1/\omega)$ compares the building's relaxation time $1/k$ with the time scale of the forcing, $1/\omega = 24\text{ h}/(2\pi) \approx 3.8$ h.
(c) $\Omega \ll 1$*:* the outside changes slowly compared with how fast the building responds, so the building keeps up: $\theta \approx \cos(\Omega s)$ and the inside follows the outside. $\Omega \gg 1$*:* the forcing reverses many times within one relaxation time. Over a short interval it pushes θ by only about $\int \cos(\Omega s)\,\mathrm{d}s \sim 1/\Omega$ before reversing, so the pushes nearly cancel and θ stays close to 0: the inside sits near the daily mean $\bar{T}$, with a small wiggle of size about $1/\Omega$.
(In Chapter 4 you will be able to solve the ODE with an integrating factor. The long-time response has amplitude $1/\sqrt{1+\Omega^2}$, which agrees with both limits.)
(d) Shed: $\Omega = 0.262/1 \approx 0.26$, so it tracks the outside, swinging by nearly the full $\pm 8\,^\circ$C (about $\pm 7.7\,^\circ$C, slightly delayed). Cellar: $1/k = 240$ h, so $\Omega \approx 0.262 \times 240 \approx 63$, and the swing is reduced by a factor of about 63, to roughly $\pm 0.1\,^\circ$C. This is why cellars stay at an almost constant temperature.
(e) Both satisfy the same ODE, so subtracting,

$$\frac{\mathrm{d}w}{\mathrm{d}s} = -w \quad \Longrightarrow \quad w = w(0)\,e^{-s}.$$

The difference decays, so after a few relaxation times (s of about 3 to 5) the two buildings behave identically. The group θ_0 only affects a transient; the long-term behaviour depends on Ω alone. (For the cellar, "a few relaxation times" is several weeks.)

Question 4

Falling with quadratic drag. Question 7 of Workshop sheet 1 suggested replacing linear drag with quadratic drag for larger objects:

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - cv^2, \qquad v(0) = 0.$$

(a) Find the dimensions of c, and show that the terminal velocity is $v_T = \sqrt{mg/c}$.

(b) Let $V = v/v_T$ and $S = t/T$. Show that choosing $T = v_T/g$ gives

$$\frac{\mathrm{d}V}{\mathrm{d}S} = 1 - V^2, \qquad V(0) = 0,$$

an equation with no parameters at all. Interpret T physically.

(c) A skydiver has $m = 80$ kg and $c = 0.25\ \mathrm{kg\,m^{-1}}$. Evaluate v_T and T. If the mass were doubled with c unchanged, by what factors would v_T and T change?

(d) For linear drag ($m\,\mathrm{d}v/\mathrm{d}t = mg - bv$, sheet 1 Question 2), the same procedure gives $\mathrm{d}V/\mathrm{d}S = 1 - V$. Without solving either equation, show that in dimensionless terms the quadratic-drag faller always approaches terminal speed at least as fast as the linear-drag one. (Hint: write $1 - V^2 = (1 - V)(1 + V)$.) Sketch both curves $V(S)$ on the same axes.

Solution

(a) $[cv^2]$ is a force, so $[c] = \mathrm{M\,L\,T^{-2}/(L^2T^{-2})} = \mathrm{M\,L^{-1}}$. At terminal velocity $\mathrm{d}v/\mathrm{d}t = 0$, so $mg = cv_T^2$ and $v_T = \sqrt{mg/c}$.
(b) With $v = v_T V$ and $t = TS$,

$$\frac{mv_T}{T}\frac{\mathrm{d}V}{\mathrm{d}S} = mg - cv_T^2V^2 = mg(1 - V^2).$$

Choosing $T = v_T/g$ makes the left-hand coefficient mg, so $\mathrm{d}V/\mathrm{d}S = 1 - V^2$. $T = v_T/g = \sqrt{m/(gc)}$ is the time a body falling freely, with no drag, would take to reach the terminal speed: the natural time for drag to matter.
(c) $v_T = \sqrt{80 \times 9.81/0.25} \approx 56\ \mathrm{m\,s^{-1}}$ (about $200\ \mathrm{km\,h^{-1}}$) and $T = v_T/g \approx 5.7$ s. Since $v_T \propto \sqrt{m}$ and $T \propto \sqrt{m}$, doubling the mass multiplies both by $\sqrt{2} \approx 1.41$. The dimensionless curve $V(S)$ is unchanged: every fall collapses onto it.
(d) $1 - V^2 = (1 - V)(1 + V) \geq 1 - V$ for $0 \leq V \leq 1$, so at any given V the quadratic-drag faller is accelerating at least as fast. Both curves start at $V = 0$ with slope 1, both rise towards $V = 1$, and the quadratic one lies above the linear one throughout. (In Chapter 4 you will solve them: $V = 1 - e^{-S}$ and $V = \tanh S$.) Note that the linear-drag time scale m/b from sheet 1 is also v_T/g, so the comparison is fair.

Question 5

Collapsing data. Two cups of tea cool in different rooms. Cup A starts at 90 °C in a room at 20 °C; cup B starts at 70 °C in a cooler room at 15 °C. The measured temperatures are:

Cup A: t (min)	0	2	4	6	8	10	15	20
T (°C)	90.0	77.5	67.5	59.0	51.0	46.0	35.5	29.0
Cup B: t (min)	0	4	8	12	16	20	30	40
T (°C)	70.0	60.5	52.0	45.5	40.0	35.0	27.5	22.5

(a) Plot T against t for both cups on the same axes.

(b) For each reading, compute the dimensionless temperature $\theta = (T - T_\infty)/(T_0 - T_\infty)$.

(c) Assuming Newton's law of cooling, estimate the cooling rate k for each cup, either from the slope of $\ln\theta$ against t or from the time taken for θ to halve.

(d) Using your values of k, plot θ against $s = kt$ for both cups on the same axes, and add the curve e^{-s}. What do you see?

Solution

(a) Two different decaying curves: cup A starts hotter, falls faster and levels off towards 20 °C; cup B falls more slowly towards 15 °C.
(b) Cup A has $T_0 - T_\infty = 70$, cup B has $T_0 - T_\infty = 55$.

Cup A: θ	1	0.821	0.679	0.557	0.443	0.371	0.221	0.129
Cup B: θ	1	0.827	0.673	0.555	0.455	0.364	0.227	0.136

(c) Newton's law gives $\theta = e^{-kt}$, so $\ln\theta = -kt$, a straight line through the origin. Cup A: $\ln\theta(20) \approx -2.05$, giving $k \approx 0.10$ min^{-1} (a least-squares fit gives 0.102). Equivalently, θ halves between $t = 6$ and 8, at about 7 minutes, and $k = \ln 2/7 \approx 0.099$. Cup B: $\ln\theta(40) \approx -1.99$, giving $k \approx 0.05$ min^{-1}; θ halves at about 14 minutes.
(d) With $k_A = 0.10$ and $k_B = 0.05$, both sets of readings are at $s = 0, 0.2, 0.4, 0.6, 0.8, 1.0, 1.5, 2.0$, and the θ values in the table in (b) agree to within the measurement noise. The two curves collapse onto the single curve e^{-s}.

Question 6

A first dimension matrix. (Attempt this if Section 3.2 has been covered in lectures.) A mass m on a spring of stiffness κ (force per unit extension) oscillates with amplitude A. We want to know how its period τ depends on m, κ and A.

(a) Write down the dimensions of τ, m, κ and A.

(b) Construct the dimension matrix, find the number of dimensionless groups, and solve for the exponents to show that $\tau = C\sqrt{m/\kappa}$ for some dimensionless constant C.

(c) Explain, by looking at the dimension matrix, why the amplitude A cannot appear in the answer.

(d) **[Extension]** For a pendulum, Section 3.1 of the notes set aside the swing amplitude α. Including it, show that dimensional analysis gives $\tau = \sqrt{l/g}\,F(\alpha)$ for some unknown function F. Why is the amplitude allowed to appear here but not for the spring?

Solution

(a) $[\tau] = \mathrm{T}$, $[m] = \mathrm{M}$, $[\kappa] = \mathrm{M\,L\,T^{-2}/L} = \mathrm{M\,T^{-2}}$, $[A] = \mathrm{L}$.
(b) With $\Pi = \tau^{k_1} m^{k_2} \kappa^{k_3} A^{k_4}$:

	τ (k_1)	m (k_2)	κ (k_3)	A (k_4)
M	0	1	1	0
L	0	0	0	1
T	1	0	−2	0

Here $n = 4$ and $k = 3$ (the matrix has rank 3), so there is $4 - 3 = 1$ group. The rows give $k_2 + k_3 = 0$, $k_4 = 0$ and $k_1 - 2k_3 = 0$. Taking $k_1 = 1$: $k_3 = \frac{1}{2}$, $k_2 = -\frac{1}{2}$, $k_4 = 0$, so $\Pi = \tau\sqrt{\kappa/m}$, and $\Pi = C$ gives $\tau = C\sqrt{m/\kappa}$. (Solving the ODE $m\ddot{x} = -\kappa x$ gives $C = 2\pi$.)
(c) A is the only parameter with a length dimension, so the L row reads $k_4 = 0$: nothing else could cancel the length in A. The period of a linear spring is independent of the amplitude.
(d) The parameters are now τ, l, g, m and α, with α already dimensionless. So $n = 5$, $k = 3$, and there are two groups: $\tau\sqrt{g/l}$ (found in Section 3.2) and α itself. The theorem gives $\Phi(\tau\sqrt{g/l}, \alpha) = 0$, i.e. $\tau = \sqrt{l/g}\, F(\alpha)$. The amplitude of a pendulum is an angle, which is already dimensionless, so it can enter freely; the amplitude of the spring carries units of length that nothing else can cancel. Dimensional analysis cannot find F: small-angle theory gives $F(0) = 2\pi$, and in fact F grows with α (by about 7% at $\alpha = 60°$).