MATH4120: Mathematical Modelling and Programming

Workshop sheet 1: Modelling practice

Chapter 1: dimensions, first-order ODE models and difference equations

NOT ASSESSED
Week 2 workshops

This sheet is not assessed. It accompanies the Week 2 workshop and gives further practice with the Chapter 1 material, alongside the exercises at the end of Chapter 1. Work through as much as you can in the workshop and finish the rest in your own time. Question 7 is designed for discussion in small groups.

Question 1

Dimensions and gravity. Newton's law of gravitation gives the attractive force between masses m_1 and m_2 a distance r apart as

$$F = \frac{G m_1 m_2}{r^2}.$$

(a) Find the dimensions of the gravitational constant G.

(b) The *escape velocity* from the surface of a planet of mass M and radius r is $v = \sqrt{2GM/r}$. Check that this formula is dimensionally homogeneous.

(c) Kepler's third law states that a planet orbiting a star of mass M with orbital radius a has period T satisfying $T^2 = 4\pi^2 a^3/(GM)$. Check this formula too.

(d) A student misremembers the escape velocity as $v = \sqrt{GM/r}$. Would a dimensional check catch the mistake? What does this tell you about what dimensional analysis can and cannot do?

Question 2

Falling with drag. A small particle of mass m falls from rest through still air. Gravity pulls it down with force mg, and air resistance opposes the motion with a force bv, where $v(t)$ is its downward speed and b is a positive constant.

(a) Use Newton's second law to write down an ODE for $v(t)$. Find the dimensions of b and check that your equation is dimensionally homogeneous.

(b) Solve the ODE with $v(0) = 0$.

(c) Find the *terminal velocity* v_T, and show that the combination $\tau = m/b$ has dimensions of time. How long does the particle take to reach 99% of v_T, in terms of τ?

(d) For a small sphere of radius R and density ρ, the drag constant is $b = 6\pi\mu R$, where μ is the viscosity of air. A fog droplet has $R = 10\ \mu$m, $\rho = 1000\ \mathrm{kg\,m^{-3}}$, and $\mu = 1.8 \times 10^{-5}\ \mathrm{Pa\,s}$. Take $g = 9.81\ \mathrm{m\,s^{-2}}$. Show that $v_T = 2\rho g R^2/(9\mu)$ and evaluate it, and evaluate τ. Roughly how long does fog take to settle through 100 m?

(e) Use the same formula for a raindrop of radius 1 mm. Real raindrops of this size fall at a few metres per second. What does this tell you about the model?

Question 3

Draining a conical tank. Section 1.5 of the notes drains a cylindrical tank. Here the tank is

an inverted cone with a small hole of area a at its tip, shaped so that when the water is h deep, the water surface is a circle of radius h. Torricelli's law still gives outflow rate $a\sqrt{2gh}$.

(a) Explain why the volume of water satisfies $\pi h^2 \dfrac{\mathrm{d}h}{\mathrm{d}t} = -a\sqrt{2gh}$. (Hint: when the depth changes by a small amount δh, roughly what volume is gained or lost?)

(b) Solve this ODE with $h(0) = h_0$, and show that the tank empties at

$$t_{\text{drain}} = \frac{2\pi h_0^{5/2}}{5a\sqrt{2g}}.$$

(c) Evaluate t_{drain} for $h_0 = 0.5$ m and $a = 1.0 \times 10^{-4}$ m^2.

(d) For the cylinder, doubling the initial depth multiplies the drain time by $\sqrt{2}$. What is the factor for the cone? Explain physically why the two shapes behave so differently.

Question 4

A tank with an inflow. A cylindrical tank of cross-sectional area A has a hole of area a in its base. Water is pumped in at a constant rate Q (in m^3 s^{-1}), and flows out according to Torricelli's law.

(a) Write down a word equation and hence an ODE for the depth $h(t)$.

(b) Find the equilibrium (steady-state) depth h^*, and check that your expression has dimensions of length.

(c) Without solving the ODE, explain why the depth moves towards h^* whether it starts above or below it.

(d) Evaluate h^* for $Q = 1.0 \times 10^{-3}$ m^3 s^{-1} and $a = 5.0 \times 10^{-4}$ m^2. If the hole is partly blocked so that its area halves, what happens to h^*?

(e) Notice that h^* does not depend on A. What *does* A affect?

Question 5

Harvesting a fish population. The fish in a lake increase by 10% each year through breeding. At the end of each year, after breeding, 1200 fish are caught. Let P_n be the number of fish after n years, with $P_0 = 10{,}000$.

(a) Write down a difference equation for P_n in the form $P_{n+1} = aP_n + b$.

(b) Find the equilibrium P^* and determine its stability.

(c) Find P_n explicitly, and hence the year in which the population collapses.

(d) What is the largest annual catch that the lake can sustain indefinitely, starting from 10,000 fish? Interpret your answer.

(e) According to this model, what happens if the catch is set slightly below your answer to (d)? Is that realistic?

Question 6

A cobweb market. Farmers decide how much wheat to plant based on *last* year's price, but the price this year is set by how much wheat is actually available. Suppose that in year n, with price p_n, the quantity demanded and the quantity supplied (in suitable units) are

$$D_n = 100 - 2p_n, \qquad S_n = 20 + 1.5\,p_{n-1}.$$

(a) Assuming the market clears each year ($D_n = S_n$), show that $p_n = 40 - 0.75\, p_{n-1}$.

(b) Find the equilibrium price p^*.

(c) Starting from $p_0 = 10$, compute $p_1, \ldots, p_4$. Describe what happens, and explain the behaviour using the value of a.

(d) Repeat the derivation in (a) with general demand $D_n = \alpha - \beta p_n$ and supply $S_n = \gamma + \delta p_{n-1}$, where all four constants are positive. Show that the price settles down if $\delta < \beta$. Interpret this condition in words.

Question 7

Group discussion: critiquing a model. In small groups, choose either the falling droplet of Question 2 or the fishery of Question 5.

(a) List three assumptions built into the model.

(b) Which assumption do you expect to fail first, and under what conditions? How could you test this with data or a simple experiment?

(c) Propose a refined model. Write down the new equation, and say what new parameters it introduces and how they could be measured. You do not need to solve it.