MATH4120: Mathematical Modelling and Programming

Workshop sheet 1: Modelling practice

Chapter 1: dimensions, first-order ODE models and difference equations

NOT ASSESSED
Week 2 workshops

Model solutions, do not circulate until after the workshops.

This sheet is not assessed. It accompanies the Week 2 workshop and gives further practice with the Chapter 1 material, alongside the exercises at the end of Chapter 1. Work through as much as you can in the workshop and finish the rest in your own time. Question 7 is designed for discussion in small groups.

Question 1

Dimensions and gravity. Newton's law of gravitation gives the attractive force between masses m_1 and m_2 a distance r apart as

$$F = \frac{Gm_1m_2}{r^2}.$$

(a) Find the dimensions of the gravitational constant G.

(b) The *escape velocity* from the surface of a planet of mass M and radius r is $v = \sqrt{2GM/r}$. Check that this formula is dimensionally homogeneous.

(c) Kepler's third law states that a planet orbiting a star of mass M with orbital radius a has period T satisfying $T^2 = 4\pi^2 a^3/(GM)$. Check this formula too.

(d) A student misremembers the escape velocity as $v = \sqrt{GM/r}$. Would a dimensional check catch the mistake? What does this tell you about what dimensional analysis can and cannot do?

Solution

(a) Rearranging, $G = Fr^2/(m_1m_2)$, so

$$[G] = \frac{\mathrm{M\,L\,T^{-2}\cdot L^2}}{\mathrm{M^2}} = \mathrm{M^{-1}\,L^3\,T^{-2}}.$$

(b) $[GM/r] = \mathrm{M^{-1}L^3T^{-2}\cdot M\cdot L^{-1}} = \mathrm{L^2\,T^{-2}}$, so $[\sqrt{2GM/r}] = \mathrm{L\,T^{-1}}$, a speed. ✓
(c) $[a^3/(GM)] = \mathrm{L^3/(M^{-1}L^3T^{-2}\cdot M)} = \mathrm{T^2} = [T^2]$. ✓
(d) No. The wrong formula differs only by the dimensionless factor $\sqrt{2}$, so it has the right dimensions. Dimensional analysis can show that a formula is *wrong*, but passing the check does not show it is right: it cannot detect missing dimensionless constants such as 2, π or $\sqrt{2}$.

Question 2

Falling with drag. A small particle of mass m falls from rest through still air. Gravity pulls it down with force mg, and air resistance opposes the motion with a force bv, where $v(t)$ is its downward speed and b is a positive constant.

(a) Use Newton's second law to write down an ODE for $v(t)$. Find the dimensions of b and check that your equation is dimensionally homogeneous.

(b) Solve the ODE with $v(0) = 0$.

(c) Find the *terminal velocity* v_T, and show that the combination $\tau = m/b$ has dimensions of time. How long does the particle take to reach 99% of v_T, in terms of τ?

(d) For a small sphere of radius R and density ρ, the drag constant is $b = 6\pi\mu R$, where μ is the viscosity of air. A fog droplet has $R = 10\ \mu$m, $\rho = 1000\ \mathrm{kg\,m^{-3}}$, and $\mu = 1.8 \times 10^{-5}$ Pa s. Take $g = 9.81\ \mathrm{m\,s^{-2}}$. Show that $v_T = 2\rho g R^2/(9\mu)$ and evaluate it, and evaluate τ. Roughly how long does fog take to settle through 100 m?

(e) Use the same formula for a raindrop of radius 1 mm. Real raindrops of this size fall at a few metres per second. What does this tell you about the model?

Solution

(a) Mass × acceleration = net downward force:

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - bv.$$

Since $[bv]$ must be a force, $[b] = \mathrm{M\,L\,T^{-2}/L\,T^{-1}} = \mathrm{M\,T^{-1}}$. Every term then has dimensions $\mathrm{M\,L\,T^{-2}}$. ✓

(b) Separating, $\displaystyle\int \frac{\mathrm{d}v}{g - (b/m)v} = \int \mathrm{d}t$, so $-\frac{m}{b}\ln\left|g - \frac{b}{m}v\right| = t + c$. Rearranging and using $v(0) = 0$,

$$v(t) = \frac{mg}{b}\left(1 - e^{-bt/m}\right).$$

(c) As $t \to \infty$, $v \to v_T = mg/b$. We have $[m/b] = \mathrm{M/M\,T^{-1}} = \mathrm{T}$, and $v = v_T(1 - e^{-t/\tau})$. Setting $v = 0.99\,v_T$ gives $e^{-t/\tau} = 0.01$, so $t = \tau \ln 100 \approx 4.6\,\tau$.

(d) With $m = \frac{4}{3}\pi R^3\rho$,

$$v_T = \frac{\frac{4}{3}\pi R^3 \rho g}{6\pi\mu R} = \frac{2\rho g R^2}{9\mu} = \frac{2 \cdot 1000 \cdot 9.81 \cdot (10^{-5})^2}{9 \cdot 1.8 \times 10^{-5}} \approx 0.012\ \mathrm{m\,s^{-1}},$$

and $\tau = m/b = v_T/g \approx 1.2 \times 10^{-3}$ s. The droplet reaches terminal speed within a few milliseconds, so it effectively falls at 1.2 $\mathrm{cm\,s^{-1}}$ throughout. Falling 100 m takes about $100/0.0121 \approx 8300$ s, i.e. over two hours. This is why fog hangs in the air.

(e) $v_T \propto R^2$, so increasing R by a factor of 100 multiplies v_T by 10^4, giving about 120 $\mathrm{m\,s^{-1}}$. This is far too fast. The linear drag law $F = bv$ only holds for very small, slow objects; larger drops need a different drag law (one that grows faster with speed). The model is fine for fog but fails for rain, and the failure shows up first in the interpret/validate stage of the modelling cycle.

Question 3

Draining a conical tank. Section 1.5 of the notes drains a cylindrical tank. Here the tank is an inverted cone with a small hole of area a at its tip, shaped so that when the water is h deep, the water surface is a circle of radius h. Torricelli's law still gives outflow rate $a\sqrt{2gh}$.

(a) Explain why the volume of water satisfies $\pi h^2 \dfrac{\mathrm{d}h}{\mathrm{d}t} = -a\sqrt{2gh}$. (Hint: when the depth changes by a small amount δh, roughly what volume is gained or lost?)

(b) Solve this ODE with $h(0) = h_0$, and show that the tank empties at

$$t_{\text{drain}} = \frac{2\pi h_0^{5/2}}{5a\sqrt{2g}}.$$

(c) Evaluate t_{drain} for $h_0 = 0.5$ m and $a = 1.0 \times 10^{-4}$ m^2.

(d) For the cylinder, doubling the initial depth multiplies the drain time by $\sqrt{2}$. What is the factor for the cone? Explain physically why the two shapes behave so differently.

Solution

(a) A thin layer of water of thickness δh at depth h is approximately a disc of area πh^2, so $\delta V \approx \pi h^2\,\delta h$ and $\mathrm{d}V/\mathrm{d}t = \pi h^2\,\mathrm{d}h/\mathrm{d}t$. Conservation of volume says this equals minus the outflow rate. (Alternatively, differentiate $V = \frac{1}{3}\pi h^3$ using the chain rule.)
(b) Write $c = a\sqrt{2g}/\pi$. Then $h^2\,\mathrm{d}h/\mathrm{d}t = -c\,h^{1/2}$, which separates as

$$\int h^{3/2}\,\mathrm{d}h = -\int c\,\mathrm{d}t \quad\Longrightarrow\quad \tfrac{2}{5}h^{5/2} = \tfrac{2}{5}h_0^{5/2} - ct.$$

So $h(t) = \left(h_0^{5/2} - \frac{5}{2}ct\right)^{2/5}$, which reaches zero when

$$t_{\text{drain}} = \frac{2h_0^{5/2}}{5c} = \frac{2\pi h_0^{5/2}}{5a\sqrt{2g}}.$$

(c) $\sqrt{2g} \approx 4.43$ and $h_0^{5/2} = 0.5^{2.5} \approx 0.177$, so $t_{\text{drain}} \approx \dfrac{2\pi \cdot 0.177}{5 \cdot 10^{-4} \cdot 4.43} \approx 500$ s, about $8\frac{1}{2}$ minutes.
(d) $t_{\text{drain}} \propto h_0^{5/2}$, so doubling h_0 multiplies the drain time by $2^{5/2} \approx 5.7$. Doubling the depth of a cone multiplies its volume by 8, and most of that extra water sits in the wide upper part. In the cylinder, doubling the depth only doubles the volume, while the extra pressure speeds up the outflow.
Extension. A cylinder with the same initial depth and volume as the cone has $A = \frac{1}{3}\pi h_0^2$; using the notes' formula, it takes $\frac{5}{3}$ as long as the cone to drain. The cone wins because its water is mostly near the top, where the pressure driving the outflow is greatest.

Question 4

A tank with an inflow. A cylindrical tank of cross-sectional area A has a hole of area a in its base. Water is pumped in at a constant rate Q (in m^3 s^{-1}), and flows out according to Torricelli's law.

(a) Write down a word equation and hence an ODE for the depth $h(t)$.

(b) Find the equilibrium (steady-state) depth h^*, and check that your expression has dimensions of length.

(c) Without solving the ODE, explain why the depth moves towards h^* whether it starts above or below it.

(d) Evaluate h^* for $Q = 1.0 \times 10^{-3}$ m^3 s^{-1} and $a = 5.0 \times 10^{-4}$ m^2. If the hole is partly blocked so that its area halves, what happens to h^*?

(e) Notice that h^* does not depend on A. What *does* A affect?

Solution

(a) Rate of change of volume = inflow − outflow, so

$$A\frac{\mathrm{d}h}{\mathrm{d}t} = Q - a\sqrt{2gh}.$$

(b) At equilibrium $\mathrm{d}h/\mathrm{d}t = 0$, so $a\sqrt{2gh^*} = Q$ and

$$h^* = \frac{1}{2g}\left(\frac{Q}{a}\right)^2.$$

Q/a is a speed ($\mathrm{L}^3\mathrm{T}^{-1}/\mathrm{L}^2$), so $[h^*] = \mathrm{L}^2\mathrm{T}^{-2}/\mathrm{L}\,\mathrm{T}^{-2} = \mathrm{L}$. ✓
(c) The outflow $a\sqrt{2gh}$ increases with h. If $h > h^*$, outflow exceeds Q, so $\mathrm{d}h/\mathrm{d}t < 0$ and the level falls. If $h < h^*$, outflow is less than Q, so the level rises. Either way h moves towards h^*.
(d) $Q/a = 2\,\mathrm{m\,s^{-1}}$, so $h^* = 4/(2\cdot 9.81) \approx 0.20$ m. Since $h^* \propto 1/a^2$, halving a multiplies h^* by 4, to about 0.82 m. A partial blockage can easily overflow a tank that was safe before.
(e) The area A sets how *fast* the level changes: the same volume imbalance moves the level of a wide tank more slowly. So A controls the time taken to approach equilibrium, but not the equilibrium itself.

Question 5

Harvesting a fish population. The fish in a lake increase by 10% each year through breeding. At the end of each year, after breeding, 1200 fish are caught. Let P_n be the number of fish after n years, with $P_0 = 10{,}000$.

(a) Write down a difference equation for P_n in the form $P_{n+1} = aP_n + b$.

(b) Find the equilibrium P^* and determine its stability.

(c) Find P_n explicitly, and hence the year in which the population collapses.

(d) What is the largest annual catch that the lake can sustain indefinitely, starting from 10,000 fish? Interpret your answer.

(e) According to this model, what happens if the catch is set slightly below your answer to (d)? Is that realistic?

Solution

(a) $P_{n+1} = 1.1\,P_n - 1200$, so $a = 1.1$ and $b = -1200$.
(b) $P^* = b/(1-a) = -1200/(-0.1) = 12{,}000$. Since $|a| = 1.1 > 1$ the equilibrium is **unstable**: a population above 12,000 grows without bound, and one below it declines.
(c) $P_n = P^* + a^n(P_0 - P^*) = 12{,}000 - 2000\,(1.1)^n$. This reaches zero when $(1.1)^n = 6$, i.e. $n = \ln 6/\ln 1.1 \approx 18.8$, so the population collapses during year 19.
(d) With a catch of H, $P^* = 10H$, and the population does not decline provided $P_0 \geq P^*$, i.e. $H \leq 1000$. This is exactly one year's growth of the initial stock (10% of 10,000): catching more than the annual increase eats into the breeding stock.
(e) The population then grows geometrically for ever, which is clearly unrealistic. Real lakes have a carrying capacity, so growth slows as the population rises. A better model would use logistic growth with harvesting, which we study in Chapter 5.

Question 6

A cobweb market. Farmers decide how much wheat to plant based on *last* year's price, but the price this year is set by how much wheat is actually available. Suppose that in year n, with price p_n, the quantity demanded and the quantity supplied (in suitable units) are

$$D_n = 100 - 2p_n, \qquad S_n = 20 + 1.5\,p_{n-1}.$$

(a) Assuming the market clears each year ($D_n = S_n$), show that $p_n = 40 - 0.75\,p_{n-1}$.

(b) Find the equilibrium price p^*.

(c) Starting from $p_0 = 10$, compute $p_1, \ldots, p_4$. Describe what happens, and explain the behaviour using the value of a.

(d) Repeat the derivation in (a) with general demand $D_n = \alpha - \beta p_n$ and supply $S_n = \gamma + \delta p_{n-1}$, where all four constants are positive. Show that the price settles down if $\delta < \beta$. Interpret this condition in words.

Solution

(a) $100 - 2p_n = 20 + 1.5p_{n-1}$, so $2p_n = 80 - 1.5p_{n-1}$ and $p_n = 40 - 0.75\,p_{n-1}$.
(b) $p^* = 40/(1 + 0.75) = 160/7 \approx 22.86$.
(c) $p_1 = 32.5$, $p_2 \approx 15.63$, $p_3 \approx 28.28$, $p_4 \approx 18.79$. The price alternates above and below p^*, with the swings shrinking. Here $a = -0.75$: since $a < 0$ the deviation $a^n(p_0 - p^*)$ changes sign every year, and since $|a| < 1$ it decays. A high price encourages planting, which gives a glut and a low price the next year, and so on. This oscillation cannot happen in a first-order ODE model.
(d) $\alpha - \beta p_n = \gamma + \delta p_{n-1}$ gives

$$p_n = -\frac{\delta}{\beta}\,p_{n-1} + \frac{\alpha - \gamma}{\beta},$$

so $a = -\delta/\beta$. The equilibrium is stable when $|a| < 1$, i.e. $\delta < \beta$. In words: the price settles if farmers' planting responds less strongly to price than consumers' buying does. If farmers overreact ($\delta > \beta$), the swings grow year on year.

Question 7

Group discussion: critiquing a model. In small groups, choose either the falling droplet of Question 2 or the fishery of Question 5.

(a) List three assumptions built into the model.

(b) Which assumption do you expect to fail first, and under what conditions? How could you test this with data or a simple experiment?

(c) Propose a refined model. Write down the new equation, and say what new parameters it introduces and how they could be measured. You do not need to solve it.

Solution

Answers will vary; look for specific assumptions, concrete tests and a written-down equation. For example:
Droplet. Assumptions: drag is linear in v; still air (no wind or updrafts); the droplet does not evaporate or grow; buoyancy is negligible. Linear drag fails first for larger or faster

drops, as Question 2(e) shows; a test is to measure terminal speeds of drops of different sizes and check whether $v_T \propto R^2$. A refinement is quadratic drag, $m\,\mathrm{d}v/\mathrm{d}t = mg - cv^2$, giving $v_T = \sqrt{mg/c}$, where c can be found from a measured terminal speed.

Fishery. Assumptions: constant growth rate regardless of population size; a fixed catch every year; no immigration, disease or variation in conditions; breeding and catching happen at separate, fixed times. The constant growth rate fails first, at high population, and annual survey data can test whether the growth rate falls as P rises. A refinement is

$$P_{n+1} = P_n + rP_n\left(1 - \frac{P_n}{K}\right) - H,$$

which introduces a carrying capacity K that can be estimated from the population size in years without fishing.