**MATH4120: Mathematical Modelling and Programming**

# Exercise sheet 0: Revision

**NOT ASSESSED**
**Week 1 workshops**

A-Level Revision

This sheet is not assessed. It revises the A-level topics that MATH4120 relies on. Work through as much as you can in the Week 1 workshop and finish the rest in your own time. If a question feels unfamiliar, that is the topic to revise first.

## Question 1

**Exponentials and logarithms.**

(a) Solve $5e^{-0.2t} = 1$, giving $t$ exactly and to 3 significant figures.

(b) Find the smallest integer $n$ for which $1.05^n > 2$.

(c) Write $\ln\left(\dfrac{a^3 b}{\sqrt{c}}\right)$ in terms of $\ln a$, $\ln b$ and $\ln c$.

(d) A population grows according to $N = N_0 e^{rt}$ and doubles every $T$ hours. Show that $r = (\ln 2)/T$.

(e) Two quantities are related by $y = Ax^k$. When $x = 1$, $y = 3$, and when $x = 4$, $y = 24$. Find $A$ and $k$. What straight-line graph could you plot to estimate $k$ from experimental data?

*Used in: Chapters 1, 2 and 5, whenever we solve or fit exponential and power-law models. The notes write* log *for the natural logarithm* ln*.*

## Question 2

**Differentiation.** Differentiate with respect to the independent variable.

(a) $y = x^2 e^{-3x}$

(b) $y = \ln(1 + x^2)$

(c) $y = \dfrac{\sin 2t}{t}$

(d) $y = \sqrt{4 - 3x}$

(e) The *sigmoid function* is $\sigma(z) = \dfrac{1}{1 + e^{-z}}$. Show that $\sigma'(z) = \sigma(z)(1 - \sigma(z))$.

*Used in: throughout; part (e) is a key step in training neural networks (Chapter 7).*

## Question 3

**Stationary points and quadratics.** Let $f(N) = \alpha N\left(1 - \dfrac{N}{M}\right)$ for $N \geq 0$, where $\alpha$ and $M$ are positive constants.

(a) Find the stationary point of $f$, show that it is a maximum, and find the maximum value.

(b) Sketch $y = f(N)$. On the same axes, draw the horizontal line $y = H$ for a constant $H > 0$. How many times does the line meet the curve, depending on the value of $H$?

(c) Confirm your answer to (b) using the discriminant of the quadratic equation $f(N) = H$, and find its roots when they exist.

*Used in: Chapter 5, for the logistic model with harvesting.*

### Question 4

**Integration.** Find the following.

(a) $\displaystyle\int \left(3x^2 - \frac{4}{x} + e^{2x}\right) \mathrm{d}x$

(b) $\displaystyle\int x e^{-x} \, \mathrm{d}x$

(c) $\displaystyle\int \frac{x}{1+x^2} \, \mathrm{d}x$

(d) $\displaystyle\int_0^{\pi/2} \sin^2 x \cos x \, \mathrm{d}x$

(e) $\displaystyle\int \ln x \, \mathrm{d}x$ (write $\ln x = 1 \cdot \ln x$)

(f) $\displaystyle\int_0^{\pi} \sin^2 t \, \mathrm{d}t$ (use a double-angle identity)

*Used in: Chapter 4, and whenever we solve an ODE by hand.*

### Question 5

**Partial fractions.**

(a) Express $\dfrac{x+5}{(x-1)(x+2)}$ in partial fractions.

(b) Let $M > 0$ be a constant. Show that

$$\frac{M}{N(M-N)} = \frac{1}{N} + \frac{1}{M-N}.$$

(c) Hence find $\displaystyle\int \frac{M}{N(M-N)} \, \mathrm{d}N$ for $0 < N < M$, writing your answer as a single logarithm.

*Used in: Chapter 5, to solve the logistic model.*

### Question 6

**Separable differential equations.**

(a) Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = xy^2$ with $y(0) = 1$. For which values of $x$ is your solution valid?

(b) A cup of tea at $90\,^\circ\mathrm{C}$ is left in a room at $20\,^\circ\mathrm{C}$. Its temperature $\theta$ (in $^\circ\mathrm{C}$) after $t$ minutes satisfies

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -k(\theta - 20),$$

where $k$ is a positive constant. After 5 minutes the tea is at $55\,^\circ\mathrm{C}$. Find $\theta(t)$, find $k$ exactly, and find how long the tea takes to cool to $30\,^\circ\mathrm{C}$.

*Used in: Chapters 1, 2 and 4. Part (b) is Newton's law of cooling, which we nondimensionalise in Chapter 2.*

### Question 7

**Connected rates of change.**

(a) Water leaks from the tip of an inverted conical funnel at a constant rate of 3 $\text{cm}^3\,\text{s}^{-1}$. The funnel's radius is always half its depth, so when the water is $h$ cm deep its surface has radius $h/2$ cm. Show that the volume of water is $V = \pi h^3/12$, and find the rate at which the depth is falling when $h = 2$.

(b) A spherical raindrop evaporates so that its volume decreases at a rate proportional to its surface area. Show that its radius $r$ decreases at a constant rate. If the initial radius is $r_0$, how long does the drop take to evaporate completely?

*Used in: Chapter 1, for the draining tank, where a volume balance is turned into an equation for the depth.*

### Question 8

**Sequences and series.**

(a) A sequence is defined by $u_{n+1} = \frac{1}{2}u_n + 3$ with $u_0 = 1$. Calculate $u_1, u_2, u_3, u_4$. Assuming the sequence converges to a limit $L$, find $L$.

(b) Let $v_n = u_n - L$. Show that $v_{n+1} = \frac{1}{2}v_n$, and hence find a formula for $u_n$.

(c) Find $\displaystyle\sum_{k=0}^{9} 2(1.1)^k$ to 2 decimal places, and $\displaystyle\sum_{k=0}^{\infty} 3\left(\tfrac{1}{2}\right)^k$. Compare the second answer with $L$.

*Used in: Chapter 1, for difference equations (discrete-time models).*

### Question 9

**Iteration and numerical methods.** Let $f(x) = x^3 - x - 1$.

(a) Show that $f(x) = 0$ has a root $\alpha$ between 1 and 2.

(b) Use the iteration $x_{n+1} = (x_n + 1)^{1/3}$ with $x_0 = 1.5$ to find $\alpha$ to 4 decimal places.

(c) The equation can also be rearranged as $x = x^3 - 1$. Try the iteration $x_{n+1} = x_n^3 - 1$ with $x_0 = 1.5$. What happens? Explain the difference between the two iterations, using a staircase or cobweb diagram, or the derivative of the iteration function at $\alpha$.

(d) Apply two steps of the Newton–Raphson method to $f$, starting from $x_0 = 1.5$.

*Used in: Chapter 1 and Chapter 5. The condition in (c) reappears as the stability test for equilibria of difference equations.*

### Question 10

**Trigonometry and oscillations.**

(a) Express $3\cos t + 4\sin t$ in the form $R\cos(t - \beta)$, where $R > 0$ and $0 < \beta < \frac{\pi}{2}$, giving $\beta$ in radians to 3 significant figures.

(b) Hence state the maximum value of $3\cos t + 4\sin t$ and the smallest positive $t$ at which it occurs.

(c) Show that $x(t) = 3\cos 2t + 4\sin 2t$ satisfies $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} = -4x$. What is the period of $x(t)$?

*Used in: Chapter 4, for simple harmonic motion and second-order ODEs.*

### Question 11

**Approximations for small quantities.**

(a) Find the binomial expansion of $(1 + x)^{1/2}$ up to and including the term in $x^2$, and state the values of $x$ for which the full expansion is valid.

(b) Use your expansion to estimate $\sqrt{1.02}$, and compare with your calculator.

(c) Using the small-angle approximations $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \frac{1}{2}\theta^2$, find the approximate value of $\dfrac{1-\cos\theta}{\theta\sin\theta}$ when $\theta$ is small.

(d) A pendulum's angle $\theta(t)$ satisfies $\dfrac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\dfrac{g}{\ell}\sin\theta$. Write down the approximate equation for small $\theta$, and compare it with the equation in Question 10(c).

*Used in: Chapters 3 and 4 (the pendulum) and Chapter 5 (linearising about an equilibrium).*

### Question 12

**Bonus: differentiating under the integral sign.** This question goes beyond A-level. The aim is to evaluate

$$\int_0^1 \frac{x-1}{\ln x}\,\mathrm{d}x.$$

(a) Explain why the integrand tends to 0 as $x \to 0^+$. By writing $x = 1+u$ and using $\ln(1+u) \approx u$ for small $u$, show that it tends to 1 as $x \to 1$. (So the integrand is well behaved on $[0,1]$ and the integral makes sense.)

(b) For $a \geq 0$ define

$$I(a) = \int_0^1 \frac{x^a - 1}{\ln x}\,\mathrm{d}x.$$

Write down $I(0)$.

(c) You may use the rule

$$\frac{\mathrm{d}}{\mathrm{d}a}\int_0^1 f(x,a)\,\mathrm{d}x = \int_0^1 \frac{\partial f}{\partial a}\,\mathrm{d}x,$$

where $\partial f/\partial a$ means differentiating $f$ with respect to $a$ while treating $x$ as a constant. Using $\dfrac{\mathrm{d}}{\mathrm{d}a}\,x^a = x^a \ln x$, show that $I'(a) = \dfrac{1}{a+1}$.

(d) Hence find $I(a)$, and deduce the value of the original integral. What is $\displaystyle\int_0^1 \frac{x^2-1}{\ln x}\,\mathrm{d}x$?

(e) Check your answer numerically with the trapezium rule using four strips. Is your estimate too high or too low, and why?

*Used in: nowhere directly, but the idea of turning a hard problem into an easier one by introducing a parameter recurs throughout applied mathematics.*