MATH4120: Mathematical Modelling and Programming

Exercise sheet 0: Revision

A-Level Revision

NOT ASSESSED
Week 1 workshops

Model solutions, do not circulate until after the workshops.

This sheet is not assessed. It revises the A-level topics that MATH4120 relies on. Work through as much as you can in the Week 1 workshop and finish the rest in your own time. If a question feels unfamiliar, that is the topic to revise first.

Question 1

Exponentials and logarithms.

(a) Solve $5e^{-0.2t} = 1$, giving t exactly and to 3 significant figures.

(b) Find the smallest integer n for which $1.05^n > 2$.

(c) Write $\ln\left(\dfrac{a^3 b}{\sqrt{c}}\right)$ in terms of $\ln a$, $\ln b$ and $\ln c$.

(d) A population grows according to $N = N_0 e^{rt}$ and doubles every T hours. Show that $r = (\ln 2)/T$.

(e) Two quantities are related by $y = Ax^k$. When $x = 1$, $y = 3$, and when $x = 4$, $y = 24$. Find A and k. What straight-line graph could you plot to estimate k from experimental data?

Used in: Chapters 1, 2 and 5, whenever we solve or fit exponential and power-law models. The notes write log *for the natural logarithm* ln*.*

Solution

(a) $e^{-0.2t} = \frac{1}{5}$, so $-0.2t = \ln\frac{1}{5} = -\ln 5$ and $t = 5\ln 5 \approx 8.05$.
(b) Taking logs, $n\ln 1.05 > \ln 2$, so $n > \ln 2/\ln 1.05 \approx 14.2$ (the inequality keeps its direction because $\ln 1.05 > 0$). The smallest integer is $n = 15$.
(c) $3\ln a + \ln b - \frac{1}{2}\ln c$.
(d) $N(T) = 2N_0$ gives $e^{rT} = 2$, so $rT = \ln 2$ and $r = (\ln 2)/T$.
(e) $x = 1$ gives $A = 3$. Then $24 = 3 \cdot 4^k$, so $4^k = 8$, i.e. $2^{2k} = 2^3$ and $k = \frac{3}{2}$. Taking logs, $\ln y = \ln A + k\ln x$, so a plot of $\ln y$ against $\ln x$ is a straight line with gradient k and intercept $\ln A$.

Question 2

Differentiation. Differentiate with respect to the independent variable.

(a) $y = x^2 e^{-3x}$

(b) $y = \ln(1 + x^2)$

(c) $y = \dfrac{\sin 2t}{t}$

(d) $y = \sqrt{4 - 3x}$

(e) The *sigmoid function* is $\sigma(z) = \dfrac{1}{1+e^{-z}}$. Show that $\sigma'(z) = \sigma(z)(1-\sigma(z))$.

Used in: throughout; part (e) is a key step in training neural networks (Chapter 7).

Solution

(a) Product rule: $y' = 2xe^{-3x} - 3x^2e^{-3x} = x(2-3x)e^{-3x}$.

(b) Chain rule: $y' = \dfrac{2x}{1+x^2}$.

(c) Quotient rule: $y' = \dfrac{2t\cos 2t - \sin 2t}{t^2}$.

(d) $y = (4-3x)^{1/2}$, so $y' = \frac{1}{2}(4-3x)^{-1/2}\cdot(-3) = -\dfrac{3}{2\sqrt{4-3x}}$.

(e) Writing $\sigma = (1+e^{-z})^{-1}$, the chain rule gives

$$\sigma'(z) = -(1+e^{-z})^{-2}\cdot(-e^{-z}) = \frac{e^{-z}}{(1+e^{-z})^2} = \frac{1}{1+e^{-z}}\cdot\frac{e^{-z}}{1+e^{-z}}.$$

The second factor is $\dfrac{(1+e^{-z})-1}{1+e^{-z}} = 1-\sigma(z)$, so $\sigma' = \sigma(1-\sigma)$. ✓

Question 3

Stationary points and quadratics. Let $f(N) = \alpha N\left(1-\dfrac{N}{M}\right)$ for $N \geq 0$, where α and M are positive constants.

(a) Find the stationary point of f, show that it is a maximum, and find the maximum value.

(b) Sketch $y = f(N)$. On the same axes, draw the horizontal line $y = H$ for a constant $H > 0$. How many times does the line meet the curve, depending on the value of H?

(c) Confirm your answer to (b) using the discriminant of the quadratic equation $f(N) = H$, and find its roots when they exist.

Used in: Chapter 5, for the logistic model with harvesting.

Solution

(a) $f'(N) = \alpha\left(1-\dfrac{2N}{M}\right) = 0$ gives $N = M/2$. Since $f''(N) = -2\alpha/M < 0$, this is a maximum, with value $f(M/2) = \alpha\cdot\dfrac{M}{2}\cdot\dfrac{1}{2} = \dfrac{\alpha M}{4}$.

(b) The graph is an inverted parabola through $(0,0)$ and $(M,0)$ with its peak at $(M/2,\ \alpha M/4)$. The line $y = H$ meets it twice if $H < \alpha M/4$, touches it once at the peak if $H = \alpha M/4$, and misses it if $H > \alpha M/4$.

(c) Multiplying $f(N) = H$ by M/α and rearranging,

$$N^2 - MN + \frac{MH}{\alpha} = 0, \qquad \Delta = M^2 - \frac{4MH}{\alpha}.$$

There are two real roots when $\Delta > 0$, i.e. $H < \alpha M/4$, one repeated root when $H = \alpha M/4$, and none when $H > \alpha M/4$, as in (b). The roots are

$$N_\pm = \frac{M}{2} \pm \sqrt{\frac{M^2}{4} - \frac{MH}{\alpha}}.$$

Question 4

Integration. Find the following.

(a) $\displaystyle\int \left(3x^2 - \frac{4}{x} + e^{2x}\right) \mathrm{d}x$

(b) $\displaystyle\int x e^{-x} \, \mathrm{d}x$

(c) $\displaystyle\int \frac{x}{1+x^2} \, \mathrm{d}x$

(d) $\displaystyle\int_0^{\pi/2} \sin^2 x \cos x \, \mathrm{d}x$

(e) $\displaystyle\int \ln x \, \mathrm{d}x$ (write $\ln x = 1 \cdot \ln x$)

(f) $\displaystyle\int_0^{\pi} \sin^2 t \, \mathrm{d}t$ (use a double-angle identity)

Used in: Chapter 4, and whenever we solve an ODE by hand.

Solution

(a) $x^3 - 4\ln|x| + \frac{1}{2}e^{2x} + c$.

(b) By parts with $u = x$, $\mathrm{d}v/\mathrm{d}x = e^{-x}$: $\displaystyle\int x e^{-x}\,\mathrm{d}x = -xe^{-x} + \int e^{-x}\,\mathrm{d}x = -(x+1)e^{-x} + c$.

(c) The numerator is half the derivative of the denominator, so the integral is $\frac{1}{2}\ln(1+x^2)+c$.

(d) Substituting $u = \sin x$, $\mathrm{d}u = \cos x\,\mathrm{d}x$: $\displaystyle\int_0^1 u^2\,\mathrm{d}u = \tfrac{1}{3}$.

(e) By parts with $u = \ln x$, $\mathrm{d}v/\mathrm{d}x = 1$: $\displaystyle x\ln x - \int x \cdot \frac{1}{x}\,\mathrm{d}x = x\ln x - x + c$.

(f) $\sin^2 t = \frac{1}{2}(1 - \cos 2t)$, so the integral is $\left[\frac{1}{2}t - \frac{1}{4}\sin 2t\right]_0^{\pi} = \frac{\pi}{2}$.

Question 5

Partial fractions.

(a) Express $\dfrac{x+5}{(x-1)(x+2)}$ in partial fractions.

(b) Let $M > 0$ be a constant. Show that

$$\frac{M}{N(M-N)} = \frac{1}{N} + \frac{1}{M-N}.$$

(c) Hence find $\displaystyle\int \frac{M}{N(M-N)}\,\mathrm{d}N$ for $0 < N < M$, writing your answer as a single logarithm.

Used in: Chapter 5, to solve the logistic model.

Solution

(a) Write $x + 5 = A(x+2) + B(x-1)$. Setting $x = 1$ gives $6 = 3A$, so $A = 2$; setting $x = -2$ gives $3 = -3B$, so $B = -1$. Hence

$$\frac{x+5}{(x-1)(x+2)} = \frac{2}{x-1} - \frac{1}{x+2}.$$

(b) Over a common denominator, $\dfrac{1}{N}+\dfrac{1}{M-N}=\dfrac{(M-N)+N}{N(M-N)}=\dfrac{M}{N(M-N)}$. ✓

(c) $\displaystyle\int\left(\frac{1}{N}+\frac{1}{M-N}\right)\mathrm{d}N=\ln N-\ln(M-N)+c=\ln\left(\frac{N}{M-N}\right)+c$. (Note the minus sign from the chain rule in $\int(M-N)^{-1}\,\mathrm{d}N=-\ln(M-N)$.)

Question 6

Separable differential equations.

(a) Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x}=xy^2$ with $y(0)=1$. For which values of x is your solution valid?

(b) A cup of tea at 90 °C is left in a room at 20 °C. Its temperature θ (in °C) after t minutes satisfies

$$\frac{\mathrm{d}\theta}{\mathrm{d}t}=-k(\theta-20),$$

where k is a positive constant. After 5 minutes the tea is at 55 °C. Find $\theta(t)$, find k exactly, and find how long the tea takes to cool to 30 °C.

Used in: Chapters 1, 2 and 4. Part (b) is Newton's law of cooling, which we nondimensionalise in Chapter 2.

Solution

(a) Separating, $\displaystyle\int y^{-2}\,\mathrm{d}y=\int x\,\mathrm{d}x$, so $-\dfrac{1}{y}=\dfrac{x^2}{2}+c$. With $y(0)=1$, $c=-1$, so

$$y=\frac{1}{1-x^2/2}=\frac{2}{2-x^2}.$$

This is valid for $-\sqrt{2}<x<\sqrt{2}$: the solution becomes infinite as $x\to\pm\sqrt{2}$. (A nonlinear ODE can "blow up" at a finite value of x even though the right-hand side looks harmless.)

(b) Separating, $\displaystyle\int\frac{\mathrm{d}\theta}{\theta-20}=-\int k\,\mathrm{d}t$, so $\ln(\theta-20)=-kt+c$ and $\theta=20+Ae^{-kt}$. From $\theta(0)=90$, $A=70$:

$$\theta(t)=20+70e^{-kt}.$$

From $\theta(5)=55$, $e^{-5k}=\frac{1}{2}$, so $k=\frac{1}{5}\ln 2\approx 0.139\ \text{min}^{-1}$. Setting $\theta=30$ gives $e^{-kt}=\frac{1}{7}$, so

$$t=\frac{\ln 7}{k}=\frac{5\ln 7}{\ln 2}\approx 14.0\ \text{minutes}.$$

Question 7

Connected rates of change.

(a) Water leaks from the tip of an inverted conical funnel at a constant rate of 3 $\text{cm}^3\,\text{s}^{-1}$. The funnel's radius is always half its depth, so when the water is h cm deep its surface has radius $h/2$ cm. Show that the volume of water is $V=\pi h^3/12$, and find the rate at which the depth is falling when $h=2$.

(b) A spherical raindrop evaporates so that its volume decreases at a rate proportional to its surface area. Show that its radius r decreases at a constant rate. If the initial radius is r_0, how long does the drop take to evaporate completely?

Used in: Chapter 1, for the draining tank, where a volume balance is turned into an equation for the depth.

Solution

(a) $V = \frac{1}{3}\pi r^2 h$ with $r = h/2$ gives $V = \frac{1}{3}\pi \cdot \frac{h^2}{4} \cdot h = \dfrac{\pi h^3}{12}$. Then $\dfrac{\mathrm{d}V}{\mathrm{d}h} = \dfrac{\pi h^2}{4}$, and by the chain rule

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}V/\mathrm{d}t}{\mathrm{d}V/\mathrm{d}h} = \frac{-3}{\pi h^2/4} = -\frac{12}{\pi h^2}.$$

At $h = 2$ this is $-3/\pi$, so the depth is falling at $3/\pi \approx 0.955$ cm s^{-1}.

(b) $V = \frac{4}{3}\pi r^3$ and the surface area is $S = 4\pi r^2$. We are told $\dfrac{\mathrm{d}V}{\mathrm{d}t} = -\lambda S$ for a constant $\lambda > 0$. By the chain rule $\dfrac{\mathrm{d}V}{\mathrm{d}t} = 4\pi r^2 \dfrac{\mathrm{d}r}{\mathrm{d}t}$, so

$$4\pi r^2 \frac{\mathrm{d}r}{\mathrm{d}t} = -\lambda \cdot 4\pi r^2 \quad \Longrightarrow \quad \frac{\mathrm{d}r}{\mathrm{d}t} = -\lambda,$$

a constant. Hence $r = r_0 - \lambda t$, and the drop vanishes at $t = r_0/\lambda$, in finite time.

Question 8

Sequences and series.

(a) A sequence is defined by $u_{n+1} = \frac{1}{2}u_n + 3$ with $u_0 = 1$. Calculate u_1, u_2, u_3, u_4. Assuming the sequence converges to a limit L, find L.

(b) Let $v_n = u_n - L$. Show that $v_{n+1} = \frac{1}{2}v_n$, and hence find a formula for u_n.

(c) Find $\displaystyle\sum_{k=0}^{9} 2(1.1)^k$ to 2 decimal places, and $\displaystyle\sum_{k=0}^{\infty} 3\left(\tfrac{1}{2}\right)^k$. Compare the second answer with L.

Used in: Chapter 1, for difference equations (discrete-time models).

Solution

(a) $u_1 = 3.5$, $u_2 = 4.75$, $u_3 = 5.375$, $u_4 = 5.6875$. A limit satisfies $L = \frac{1}{2}L + 3$, so $L = 6$.

(b) $v_{n+1} = u_{n+1} - 6 = \frac{1}{2}u_n + 3 - 6 = \frac{1}{2}(u_n - 6) = \frac{1}{2}v_n$. So (v_n) is geometric with $v_n = (\frac{1}{2})^n v_0 = -5(\frac{1}{2})^n$, and

$$u_n = 6 - 5\left(\tfrac{1}{2}\right)^n.$$

(c) Using $\sum_{k=0}^{n-1} ar^k = a\dfrac{r^n - 1}{r - 1}$,

$$\sum_{k=0}^{9} 2(1.1)^k = 2 \cdot \frac{1.1^{10} - 1}{0.1} = 20(1.1^{10} - 1) \approx 31.87.$$

For $|r| < 1$ the sum to infinity is $a/(1 - r)$, so $\sum_{k=0}^{\infty} 3(\frac{1}{2})^k = 3/(1 - \frac{1}{2}) = 6 = L$. This is no coincidence: unrolling the recurrence gives $u_n = (\frac{1}{2})^n u_0 + 3\sum_{k=0}^{n-1}(\frac{1}{2})^k$, and the first term vanishes as $n \to \infty$.

Question 9

Iteration and numerical methods. Let $f(x) = x^3 - x - 1$.

(a) Show that $f(x) = 0$ has a root α between 1 and 2.

(b) Use the iteration $x_{n+1} = (x_n + 1)^{1/3}$ with $x_0 = 1.5$ to find α to 4 decimal places.

(c) The equation can also be rearranged as $x = x^3 - 1$. Try the iteration $x_{n+1} = x_n^3 - 1$ with $x_0 = 1.5$. What happens? Explain the difference between the two iterations, using a staircase or cobweb diagram, or the derivative of the iteration function at α.

(d) Apply two steps of the Newton–Raphson method to f, starting from $x_0 = 1.5$.

Used in: Chapter 1 and Chapter 5. The condition in (c) reappears as the stability test for equilibria of difference equations.

Solution

(a) $f(1) = -1 < 0$ and $f(2) = 5 > 0$. Since f is continuous, the sign change shows there is a root in $(1, 2)$.
(b) $x_1 = 1.35721$, $x_2 = 1.33086$, $x_3 = 1.32588$, $x_4 = 1.32494$, $x_5 = 1.32476$, $x_6 = 1.32473$, so $\alpha \approx 1.3247$. (To confirm, check the sign change: $f(1.32465) < 0 < f(1.32475)$.)
(c) $x_1 = 2.375$, $x_2 \approx 12.40$, $x_3 \approx 1904$: the iteration diverges. An iteration $x_{n+1} = g(x_n)$ converges to a root α from nearby starting values when $|g'(\alpha)| < 1$, and moves away when $|g'(\alpha)| > 1$. For $g(x) = (x+1)^{1/3}$, $g'(\alpha) = \frac{1}{3}(\alpha+1)^{-2/3} \approx 0.19$. For $g(x) = x^3 - 1$, $g'(\alpha) = 3\alpha^2 \approx 5.26$. On a staircase diagram, the first iteration steps inwards towards the intersection of $y = g(x)$ with $y = x$, and the second steps outwards.
(d) $f'(x) = 3x^2 - 1$ and $x_{n+1} = x_n - \dfrac{x_n^3 - x_n - 1}{3x_n^2 - 1}$. From $x_0 = 1.5$: $x_1 = 1.347826$, $x_2 = 1.325200$. (A third step gives 1.324718, already correct to 6 decimal places.)

Question 10

Trigonometry and oscillations.

(a) Express $3\cos t + 4\sin t$ in the form $R\cos(t - \beta)$, where $R > 0$ and $0 < \beta < \frac{\pi}{2}$, giving β in radians to 3 significant figures.

(b) Hence state the maximum value of $3\cos t + 4\sin t$ and the smallest positive t at which it occurs.

(c) Show that $x(t) = 3\cos 2t + 4\sin 2t$ satisfies $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} = -4x$. What is the period of $x(t)$?

Used in: Chapter 4, for simple harmonic motion and second-order ODEs.

Solution

(a) $R\cos(t-\beta) = R\cos\beta\cos t + R\sin\beta\sin t$, so $R\cos\beta = 3$ and $R\sin\beta = 4$. Hence $R = \sqrt{3^2 + 4^2} = 5$ and $\tan\beta = \frac{4}{3}$, giving $\beta \approx 0.927$. So $3\cos t + 4\sin t = 5\cos(t - 0.927)$.
(b) The maximum value is 5, when $t - \beta = 0$, i.e. $t \approx 0.927$.
(c) $x' = -6\sin 2t + 8\cos 2t$ and $x'' = -12\cos 2t - 16\sin 2t = -4(3\cos 2t + 4\sin 2t) = -4x$.
✓ The period is $2\pi/2 = \pi$.

Question 11

Approximations for small quantities.

(a) Find the binomial expansion of $(1+x)^{1/2}$ up to and including the term in x^2, and state the values of x for which the full expansion is valid.

(b) Use your expansion to estimate $\sqrt{1.02}$, and compare with your calculator.

(c) Using the small-angle approximations $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \frac{1}{2}\theta^2$, find the approximate value of $\dfrac{1 - \cos\theta}{\theta\sin\theta}$ when θ is small.

(d) A pendulum's angle $\theta(t)$ satisfies $\dfrac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\dfrac{g}{\ell}\sin\theta$. Write down the approximate equation for small θ, and compare it with the equation in Question 10(c).

Used in: Chapters 3 and 4 (the pendulum) and Chapter 5 (linearising about an equilibrium).

Solution

(a) $(1+x)^{1/2} = 1 + \frac{1}{2}x + \dfrac{\frac{1}{2}\cdot(-\frac{1}{2})}{2!}x^2 + \cdots = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$, valid for $|x| < 1$.

(b) With $x = 0.02$: $1 + 0.01 - 0.00005 = 1.00995$. The calculator gives $1.0099505\ldots$, so the estimate is correct to 7 significant figures.

(c) $\dfrac{1 - (1 - \frac{1}{2}\theta^2)}{\theta \cdot \theta} = \dfrac{\frac{1}{2}\theta^2}{\theta^2} = \dfrac{1}{2}$.

(d) $\dfrac{\mathrm{d}^2\theta}{\mathrm{d}t^2} \approx -\dfrac{g}{\ell}\theta$. This has the same form as $x'' = -4x$ in Question 10(c), with 4 replaced by g/ℓ, so small oscillations are sinusoidal with period $2\pi\sqrt{\ell/g}$.

Question 12

Bonus: differentiating under the integral sign. This question goes beyond A-level. The aim is to evaluate

$$\int_0^1 \frac{x-1}{\ln x}\,\mathrm{d}x.$$

(a) Explain why the integrand tends to 0 as $x \to 0^+$. By writing $x = 1+u$ and using $\ln(1+u) \approx u$ for small u, show that it tends to 1 as $x \to 1$. (So the integrand is well behaved on $[0,1]$ and the integral makes sense.)

(b) For $a \geq 0$ define

$$I(a) = \int_0^1 \frac{x^a - 1}{\ln x}\,\mathrm{d}x.$$

Write down $I(0)$.

(c) You may use the rule

$$\frac{\mathrm{d}}{\mathrm{d}a}\int_0^1 f(x,a)\,\mathrm{d}x = \int_0^1 \frac{\partial f}{\partial a}\,\mathrm{d}x,$$

where $\partial f/\partial a$ means differentiating f with respect to a while treating x as a constant. Using $\dfrac{\mathrm{d}}{\mathrm{d}a}x^a = x^a \ln x$, show that $I'(a) = \dfrac{1}{a+1}$.

(d) Hence find $I(a)$, and deduce the value of the original integral. What is $\displaystyle\int_0^1 \frac{x^2-1}{\ln x}\,\mathrm{d}x$?

(e) Check your answer numerically with the trapezium rule using four strips. Is your estimate too high or too low, and why?

Used in: nowhere directly, but the idea of turning a hard problem into an easier one by introducing a parameter recurs throughout applied mathematics.

Solution

(a) As $x \to 0^+$, the numerator tends to -1 while $\ln x \to -\infty$, so the quotient tends to 0. Near $x = 1$, put $x = 1 + u$:

$$\frac{x-1}{\ln x} = \frac{u}{\ln(1+u)} \approx \frac{u}{u} = 1.$$

(b) $x^0 - 1 = 0$, so $I(0) = 0$.
(c) Differentiating with respect to a with x fixed,

$$\frac{\partial}{\partial a}\left(\frac{x^a - 1}{\ln x}\right) = \frac{x^a \ln x}{\ln x} = x^a,$$

so the problematic $\ln x$ cancels, and

$$I'(a) = \int_0^1 x^a \,\mathrm{d}x = \left[\frac{x^{a+1}}{a+1}\right]_0^1 = \frac{1}{a+1}.$$

(d) Integrating, $I(a) = \ln(a+1) + c$, and $I(0) = 0$ gives $c = 0$. So

$$I(a) = \ln(a+1), \qquad \int_0^1 \frac{x-1}{\ln x}\,\mathrm{d}x = I(1) = \ln 2 \approx 0.6931,$$

and $\displaystyle\int_0^1 \frac{x^2-1}{\ln x}\,\mathrm{d}x = I(2) = \ln 3$.
(e) With $h = 0.25$ and the end values 0 and 1 from part (a), the integrand takes the values $0,\ 0.5410,\ 0.7213,\ 0.8690,\ 1$, so

$$T = 0.25\left[\tfrac{1}{2}(0+1) + 0.5410 + 0.7213 + 0.8690\right] \approx 0.658.$$

This is too low. The graph of the integrand is concave (it rises steeply near $x = 0$ and flattens out), so each chord lies below the curve.
Remark. The approach used in part (c) needs certain conditions to be valid; here it is OK because $\partial f/\partial a = x^a$ is continuous for $x \in [0,1]$ and $a \geq 0$.