MATH4120: Mathematical Modelling and Programming

Worked solutions to the exercises

All exercises from the lecture notes, chapter by chapter

SOLUTIONS RELEASED
Chapters 1–4

Model solutions to the exercises in the lecture notes: Chapters 1–4.

These are model solutions to every exercise set in the lecture notes, chapter by chapter. Conventions:

- log denotes the natural logarithm (base e) throughout.
- Numerical answers are rounded.
- For the computational exercises (Chapters 6–7) the "solution" is a combination of the hand calculations that can be done on paper, reference Python code, and the qualitative result you should observe when you run it. Exact numbers will depend on the random seed.

Chapter 1 What is a mathematical model?

Exercise 1.1 Pendulum dimensions

Solution

With $T = 2\pi\sqrt{\ell/g}$ we need the right-hand side to have dimension of time. Since 2π is dimensionless,

$$\left[\sqrt{\frac{\ell}{g}}\right] = \left(\frac{\mathrm{L}}{\mathrm{L\,T^{-2}}}\right)^{1/2} = \left(\mathrm{T^2}\right)^{1/2} = \mathrm{T}.$$

So the formula is dimensionally homogeneous. The dimensions of g are $[g] = \mathrm{L\,T^{-2}}$ (acceleration).

Exercise 1.2 Carbon-14

Solution

Model $N(t) = N_0 e^{rt}$ with $r < 0$.

(a) Half-life: $e^{r\cdot 5730} = \frac{1}{2} \Rightarrow r = -\dfrac{\log 2}{5730} \approx -1.21 \times 10^{-4}$ yr^{-1}.

(b) Fraction after 1000 years: $\dfrac{N(1000)}{N_0} = e^{1000r} = 2^{-1000/5730} \approx 0.886$ (about 88.6%).

(c) $r < 0$ means the amount *decreases* with time: the negative sign encodes radioactive decay (loss of material), as opposed to growth.

Exercise 1.3 Newton's cooling

Solution

Let $\theta = T - T_{\text{room}}$. Since T_{room} is constant, $\dot\theta = \dot T = -k(T - T_{\text{room}}) = -k\theta$. This is separable:

$$\int \frac{\mathrm{d}\theta}{\theta} = \int -k\,\mathrm{d}t \;\Rightarrow\; \log\theta = -kt + c \;\Rightarrow\; \theta(t) = \theta_0 e^{-kt}, \qquad \theta_0 = T_0 - T_{\text{room}}.$$

Hence $T(t) = T_{\text{room}} + \theta_0 e^{-kt} = T_{\text{room}} + (T_0 - T_{\text{room}})e^{-kt}$. With $T_0 = 90°\text{C}$, $T_{\text{room}} = 20°\text{C}$: $T(t) = 20 + 70\,e^{-kt}$.

Exercise 1.4 Draining tank, $A = 1\,\text{m}^2$, $a = 5\times10^{-4}\,\text{m}^2$, $h_0 = 3\,\text{m}$

Solution

(a) Conservation of volume $A\dot h = -Q$ with Torricelli $Q = a\sqrt{2gh}$ gives

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{a}{A}\sqrt{2gh}.$$

Dimensions: $[a/A] = 1$, $[\sqrt{2gh}] = \sqrt{\text{L}\,\text{T}^{-2}\cdot\text{L}} = \text{L}\,\text{T}^{-1} = [\mathrm{d}h/\mathrm{d}t]$. ✓

(b) Write $c = (a/A)\sqrt{2g}$, so $\dot h = -c\sqrt{h}$. Separating, $2\sqrt{h} = -ct + 2\sqrt{h_0}$, hence

$$h(t) = \left(\sqrt{h_0} - \tfrac{c}{2}t\right)^2.$$

(c) Drain time (when $h = 0$):

$$t_{\text{drain}} = \frac{2\sqrt{h_0}}{c} = \frac{A}{a}\sqrt{\frac{2h_0}{g}} = 2000\sqrt{\frac{6}{9.81}} \approx 1564\text{ s} \approx 26.1\text{ min}.$$

(d) $h(t)$ is an upward parabola in t: $h''(t) = c^2/2 > 0$, so it is concave up. Physically the tank drains quickly while it is deep (large pressure head, fast outflow) and ever more slowly as $h \to 0$, so the curve flattens out towards the end — a concave-up shape.

(e) With a discharge coefficient, $Q = C_d\,a\sqrt{2gh}$, so $c \mapsto C_d\,c$ and

$$t_{\text{drain}} \mapsto \frac{t_{\text{drain}}}{C_d} = \frac{1564}{0.6} \approx 2607\text{ s} \approx 43.4\text{ min}.$$

The drain time increases by the factor $1/C_d \approx 1.67$.

Exercise 1.5 Polluted lake, $V = 10^6\,\text{m}^3$

Solution

(a) Word equation:

$$\text{rate of change of pollutant mass} = \underbrace{P}_{\text{inflow}} - \underbrace{C\,Q}_{\text{outflow}}.$$

(b) Mass $= VC$ with V constant, so $V\,\dot C = P - QC$, i.e.

$$\frac{\mathrm{d}C}{\mathrm{d}t} = \frac{P}{V} - \frac{Q}{V}\,C.$$

Assumptions: the lake is perfectly well-mixed (uniform C); V, Q, P are constant; the pollutant is conserved (no decay, sedimentation or evaporation).

(c) Linear first-order ODE. Steady state $C^* = P/Q$. With $C(0) = 0$,

$$C(t) = \frac{P}{Q}(1 - e^{-Qt/V}), \qquad C(t) \to \frac{P}{Q} \text{ as } t \to \infty.$$

(d) The concentration rises monotonically to its steady value P/Q, so the binding constraint is $P/Q \le 0.01$. With $Q = 5000$,

$$P_{\max} = 0.01 \times 5000 = 50 \text{ kg/day}.$$

(e) Likely failures: stratification / incomplete mixing, seasonal variation in Q, and pollutant that actually degrades or settles. A natural refinement is a first-order removal term, $\dot{C} = P/V - (Q/V)C - \lambda C$.

Exercise 1.6 Savings account, 3%, £500/yr, $B_0 = 2000$

Solution

(a) Interest then year-end deposit: $B_{n+1} = 1.03\, B_n + 500$.

(b) Affine recurrence with $a = 1.03$, $b = 500$. Equilibrium $B^* = \dfrac{b}{1-a} = \dfrac{500}{1-1.03} = -16\,666.67$, so

$$B_n = B^* + a^n(B_0 - B^*) = -16\,666.67 + 18\,666.67\,(1.03)^n.$$

(c) $B_{20} = -16\,666.67 + 18\,666.67\,(1.03)^{20} \approx £17\,047$.

(d) Continuous model $\dot{B} = 0.03B + 500$, $B(0) = 2000$: linear, particular solution $-500/0.03 = -16\,666.67$, so $B(t) = -16\,666.67 + 18\,666.67\, e^{0.03t}$ and

$$B(20) = -16\,666.67 + 18\,666.67\, e^{0.6} \approx £17\,346.$$

The continuous figure is about £299 higher. Two reasons: continuous compounding gives a slightly larger effective annual factor ($e^{0.03} = 1.0305 > 1.03$), and the £500 is added smoothly through the year rather than as a lump at year-end, so it earns interest sooner.

Exercise 1.7 Drug dosing, 200 mg every 8 h, 40% eliminated

Solution

(a) 60% survives between doses, so just after the n-th dose

$$D_n = 0.6\, D_{n-1} + 200, \qquad D_1 = 200.$$

(b) Steady state $D^* = \dfrac{200}{1 - 0.6} = 500$ mg.

(c) Solving, $D_n = 500 - 300\,(0.6)^{n-1}$, giving $D_1 = 200$, $D_2 = 320$, $D_3 = 392$, $D_4 = 435, \cdots \to 500$. The level never exceeds $500 < 600$, so there is no risk of toxicity, and the steady state (500) sits safely inside the window. However $D_1 = 200 < 250$: the *first* dose leaves the patient briefly below the therapeutic minimum (from D_2 onwards the level is within $[250, 600]$). The regimen is safe in steady state but a *loading dose* would remove the early under-dosing.

(d) Reaching 95% of steady state means $D_n \geq 0.95 \times 500 = 475$, i.e. $300\,(0.6)^{n-1} \leq 25$, so $(0.6)^{n-1} \leq \frac{1}{12}$, giving $n - 1 \geq \log(1/12)/\log 0.6 = 4.87$. Hence $n = 6$ doses ($D_5 \approx 461$, $D_6 \approx 477$).

Exercise 1.8 Discrete logistic $N_{n+1} = RN_n(1 - N_n/K)$

Solution

(a) With $u_n = N_n/K$ (so $N_n = Ku_n$), $Ku_{n+1} = R\,Ku_n(1-u_n)$, hence $u_{n+1} = R\,u_n(1-u_n)$.
(b) Equilibria solve $u^* = R\,u^*(1-u^*)$: either $u^* = 0$, or $1 = R(1-u^*)$ giving $u^* = 1 - \frac{1}{R}$. The latter is positive (hence admissible) precisely when $R > 1$.
(c) Iterating from $u_0 = 0.2$ for 30+ steps:

- $R = 2.0$: converges monotonically to the fixed point $u^* = 0.5$.
- $R = 3.2$: the fixed point $u^* = 0.6875$ is now unstable ($|f'(u^*)| = |R(1-2u^*)| = 1.2 > 1$); the orbit settles into a *period-2* cycle, oscillating between ≈ 0.513 and ≈ 0.799.
- $R = 3.8$: the orbit is *chaotic* — aperiodic, wandering irregularly over roughly $(0.15, 0.95)$ and extremely sensitive to u_0.

(d) Because the map advances in finite jumps it can *overshoot* the equilibrium; the resulting over-correction produces oscillations and, for large R, period-doubling and chaos. The continuous logistic ODE is a first-order autonomous equation whose solutions are monotone, so it can never oscillate or behave chaotically. Discreteness is what unlocks the richer dynamics.

Chapter 2 Dimensional analysis and nondimensionalisation

Exercise 2.1 Damped pendulum

Solution

$mL^2\ddot{\theta} + bL^2\dot{\theta} + mgL\sin\theta = 0$.
(a) θ is dimensionless, $[m] = \mathrm{M}$, $[L] = \mathrm{L}$, $[g] = \mathrm{L\,T^{-2}}$, $[t] = \mathrm{T}$. Each term must equal $[mgL\sin\theta] = \mathrm{M\,L^2\,T^{-2}}$. Matching the damping term $[bL^2\dot{\theta}] = [b]\,\mathrm{L^2\,T^{-1}}$ forces $[b] = \mathrm{M\,T^{-1}}$.
(b) The angle is already dimensionless ($u = \theta$). The natural timescale is $\tau = \sqrt{L/g}$; set $s = t/\tau$.
(c) Dividing by mL^2 gives $\ddot{\theta} + \frac{b}{m}\dot{\theta} + \frac{g}{L}\sin\theta = 0$. With $s = t/\tau$ we have $\dot{\theta} = \frac{1}{\tau}u'$, $\ddot{\theta} = \frac{1}{\tau^2}u''$, so

$$\frac{1}{\tau^2}u'' + \frac{b}{m}\frac{1}{\tau}u' + \frac{g}{L}\sin u = 0 \xrightarrow{\times\,\tau^2 = L/g} u'' + \underbrace{\frac{b}{m}\sqrt{\frac{L}{g}}}_{\gamma}u' + \sin u = 0.$$

($\frac{g}{L}\tau^2 = 1$.) Here $\gamma = \frac{b}{m}\sqrt{L/g}$ is dimensionless ($[b/m] = \mathrm{T^{-1}}$, $[\sqrt{L/g}] = \mathrm{T}$) and measures the strength of damping relative to the natural oscillation: small γ is weakly damped (many swings), large γ is heavily damped.

Exercise 2.2 Chemical relaxation $\dot{C} = -C/\tau_r + S$

Solution

(a) With $C^* = C/(S\tau_r)$ and $s = t/\tau_r$: $C = S\tau_r C^*$, $\mathrm{d}C/\mathrm{d}t = S\,\mathrm{d}C^*/\mathrm{d}s$, and the right-hand side becomes $-C/\tau_r + S = S(1 - C^*)$. Cancelling S,

$$\frac{\mathrm{d}C^*}{\mathrm{d}s} = 1 - C^*.$$

(b) Steady state: $C^* = 1$ (and $C^*(s) = 1 - (1 - C_0^*)e^{-s} \to 1$).
(c) $C^* = 1$ corresponds to the dimensional steady state $C = S\tau_r$: the equilibrium concentration is the source strength multiplied by the reaction timescale.

Chapter 3 The Buckingham Π-theorem

Exercise 3.1 Kepler's third law

Solution

The orbital period τ is to be found in terms of the orbital radius R, the mass of the Sun M, and the gravitational constant G.
(a) Rearranging $F = GMm/r^2$ gives $G = Fr^2/(Mm)$. Force has dimensions $[F] = \mathrm{M\,L\,T^{-2}}$, so

$$[G] = \frac{[F]\,[r]^2}{[M]\,[m]} = \frac{\mathrm{M\,L\,T^{-2}} \cdot \mathrm{L}^2}{\mathrm{M} \cdot \mathrm{M}} = \mathrm{M^{-1}\,L^{3}\,T^{-2}}.$$

(b) $[\tau] = \mathrm{T}$, $[R] = \mathrm{L}$, $[M] = \mathrm{M}$, and $[G] = \mathrm{M^{-1}L^{3}T^{-2}}$ from (a). There are $n = 4$ parameters (τ, R, M, G) and $k = 3$ base dimensions (M, L, T), all of which appear. The Π-theorem therefore predicts $m = n - k = 4 - 3 = 1$ dimensionless group.
(c) We seek $\Pi = \tau^{k_1} R^{k_2} M^{k_3} G^{k_4}$ with $[\Pi] = 1$. The dimension matrix is

	τ (k_1)	R (k_2)	M (k_3)	G (k_4)
M	0	0	1	−1
L	0	1	0	3
T	1	0	0	−2

giving the equations

$$\begin{aligned} \mathrm{M}: &\quad k_3 - k_4 = 0, \\ \mathrm{L}: &\quad k_2 + 3k_4 = 0, \\ \mathrm{T}: &\quad k_1 - 2k_4 = 0. \end{aligned}$$

Since we want τ in terms of the other parameters, set $k_1 = 1$. Then

$$\mathrm{T}:\ k_4 = \tfrac{1}{2}, \qquad \mathrm{M}:\ k_3 = k_4 = \tfrac{1}{2}, \qquad \mathrm{L}:\ k_2 = -3k_4 = -\tfrac{3}{2},$$

so

$$\Pi = \tau\, R^{-3/2} M^{1/2} G^{1/2} = \tau\sqrt{\frac{GM}{R^3}}.$$

With only one group, the Π-theorem gives $\Pi = C$, a dimensionless constant, so

$$\tau = C\sqrt{\frac{R^3}{GM}}.$$

(Check: $[R^3/(GM)] = \mathrm{L}^3/(\mathrm{M}^{-1}\mathrm{L}^3\mathrm{T}^{-2}\cdot\mathrm{M}) = \mathrm{T}^2$, so the right-hand side has dimensions of time.) Dimensional analysis cannot find C. Balancing gravity against the centripetal force for a circular orbit, $GMm/R^2 = m\omega^2 R$ with $\omega = 2\pi/\tau$, gives $C = 2\pi$.

(d) Squaring, $\tau^2 = C^2R^3/(GM)$. Every planet orbits the same Sun, so G, M and C are the same for all of them, and

$$\frac{\tau^2}{R^3} = \frac{C^2}{GM} = \text{constant}, \qquad \text{i.e.} \qquad \tau^2 \propto R^3.$$

(e) Dividing the result for Mars by the result for the Earth, the unknown constant $C^2/(GM)$ cancels:

$$\frac{\tau_{\text{Mars}}}{\tau_{\text{Earth}}} = \left(\frac{R_{\text{Mars}}}{R_{\text{Earth}}}\right)^{3/2} = 1.52^{3/2} = 1.52 \times \sqrt{1.52} \approx 1.52 \times 1.233 \approx 1.87.$$

So a Martian year is about 1.87 Earth years. The observed value is 1.88 years.

(f) The planet's mass m cancels from its own equation of motion: the gravitational pull on it is proportional to m, and so is its inertia (in $GMm/R^2 = m\omega^2 R$, m divides out). If we include m in the list anyway, then $n = 5$ and $k = 3$, so there are $5 - 3 = 2$ groups, for example $\tau\sqrt{GM/R^3}$ and m/M, and the Π-theorem gives

$$\tau = \sqrt{\frac{R^3}{GM}}\, F\!\left(\frac{m}{M}\right)$$

for some unknown function F. (The exact two-body result is $F = 2\pi/\sqrt{1 + m/M}$, because the Sun also moves slightly.) For every planet m/M is tiny, about 3×10^{-6} for the Earth and 10^{-3} for Jupiter, so F is very close to $F(0) = 2\pi$ and the planet's mass can safely be left out.

Exercise 3.2 Water waves

Solution

The period T of a water wave depends on wavelength λ, depth h, gravity g, density ρ and surface tension σ: $n = 6$ parameters.

(a) Dimensions:

$$[T] = \mathrm{T}, \quad [\lambda] = \mathrm{L}, \quad [h] = \mathrm{L}, \quad [g] = \mathrm{L}\,\mathrm{T}^{-2}, \quad [\rho] = \mathrm{M}\,\mathrm{L}^{-3}, \quad [\sigma] = \mathrm{M}\,\mathrm{T}^{-2}.$$

(Surface tension is force per unit length: $\mathrm{M}\,\mathrm{L}\,\mathrm{T}^{-2}/\mathrm{L} = \mathrm{M}\,\mathrm{T}^{-2}$.)

(b) Three base dimensions (M, L, T), so $k = 3$ and the Π-theorem predicts $m = n - k = 6 - 3 = 3$ dimensionless groups.

(c) Seeking $\Pi = T^a\lambda^b h^c g^d \rho^e \sigma^f$ dimensionless:

$$\begin{aligned} \mathrm{M}: &\quad e + f = 0, \\ \mathrm{L}: &\quad b + c + d - 3e = 0, \\ \mathrm{T}: &\quad a - 2d - 2f = 0. \end{aligned}$$

Three constraints, six exponents ⇒ three free choices, hence three groups. A convenient independent set is

$$\Pi_1 = T\sqrt{\frac{g}{\lambda}}, \qquad \Pi_2 = \frac{h}{\lambda}, \qquad \Pi_3 = \frac{\sigma}{\rho g \lambda^2}.$$

(Check, e.g. $[\Pi_3] = \dfrac{\mathrm{M\,T^{-2}}}{\mathrm{M\,L^{-3}}\cdot\mathrm{L\,T^{-2}}\cdot\mathrm{L^2}} = 1$.) The theorem therefore gives $\Phi(\Pi_1,\Pi_2,\Pi_3)=0$, i.e.

$$T = \sqrt{\frac{\lambda}{g}}\,\phi\!\left(\frac{h}{\lambda},\ \frac{\sigma}{\rho g\lambda^2}\right),$$

which reproduces the known structure of the water-wave dispersion relation: Π_2 distinguishes deep ($h \gg \lambda$) from shallow water, and Π_3 (an inverse Bond number) measures the importance of surface tension. The choice of groups is not unique — any independent products of powers are equally valid.

Chapter 4 Ordinary differential equations

Exercise 4.1 First-order ODEs

Solution

(a) $y' = x^2/y$, $y(0)=2$. Separable: $y\,\mathrm{d}y = x^2\,\mathrm{d}x$, so $\frac{1}{2}y^2 = \frac{1}{3}x^3 + C$. $y(0)=2 \Rightarrow C=2$, hence

$$y(x) = \sqrt{\tfrac{2}{3}x^3 + 4}.$$

(b) $y' + 2y = e^{-x}$. Linear, $\mu = e^{2x}$: $(e^{2x}y)' = e^x$, so $e^{2x}y = e^x + C$ and $y(x) = e^{-x} + Ce^{-2x}$.
(c) $y' = y\sin x$, $y(0)=1$. Separable: $\log y = -\cos x + C$. $y(0)=1 \Rightarrow C=1$, so $y(x) = e^{\,1-\cos x}$.

Exercise 4.2 Bernoulli equations

Solution

(a) $y' + y = y^2$ ($n=2$). Put $v = y^{-1}$, $v' = -y^{-2}y'$. Dividing by y^2: $-v' + v = 1$, i.e. $v' - v = -1$. With $\mu = e^{-x}$, $(e^{-x}v)' = -e^{-x}$, so $v = 1 + Ce^x$ and

$$y(x) = \frac{1}{1 + Ce^x}.$$

(b) $y' - \frac{2}{x}y = -x^2y^2$, $y(1)=1$ ($n=2$). $v = y^{-1}$ gives $v' + \frac{2}{x}v = x^2$. Integrating factor $\mu = x^2$: $(x^2v)' = x^4$, so $x^2v = \frac{1}{5}x^5 + C$, $v = \frac{1}{5}x^3 + Cx^{-2}$. Then $y = 1/v = \dfrac{5x^2}{x^5 + 5C}$; the condition $y(1)=1$ gives $v(1)=1$, so $\frac{1}{5} + C = 1$, $C = \frac{4}{5}$, and

$$y(x) = \frac{5x^2}{x^5 + 4}.$$

(c) $y' + \frac{y}{x} = x\sqrt{y}$ ($n=\frac{1}{2}$). $v = y^{1/2}$, $v' = \frac{1}{2}y^{-1/2}y'$. Dividing by $y^{1/2}$ and using $y^{-1/2}y' = 2v'$: $2v' + \frac{1}{x}v = x$, i.e. $v' + \frac{1}{2x}v = \frac{x}{2}$. With $\mu = \sqrt{x}$, $(\sqrt{x}\,v)' = \frac{1}{2}x^{3/2}$, so $\sqrt{x}\,v = \frac{1}{5}x^{5/2} + C$, $v = \frac{1}{5}x^2 + Cx^{-1/2}$. Since $v = \sqrt{y}$,

$$y(x) = \left(\frac{x^2}{5} + \frac{C}{\sqrt{x}}\right)^2.$$

Exercise 4.3 Homogeneous equations

Solution

(a) $y' = \dfrac{y^2+xy}{x^2} = v^2+v$ with $v = y/x$. Then $v + xv' = v^2+v$, so $xv' = v^2$, giving $-1/v = \log|x| + C$. Hence $v = -1/(\log|x|+C)$ and

$$y(x) = \frac{-x}{\log|x|+C}.$$

(b) $y' = \dfrac{x+y}{x-y} = \dfrac{1+v}{1-v}$. Then $xv' = \dfrac{1+v}{1-v} - v = \dfrac{1+v^2}{1-v}$, so $\dfrac{1-v}{1+v^2}\,\mathrm{d}v = \dfrac{\mathrm{d}x}{x}$. Integrating, $\arctan v - \frac{1}{2}\log(1+v^2) = \log|x| + C$. Replacing $v = y/x$ and simplifying $\frac{1}{2}\log(1+y^2/x^2) = \frac{1}{2}\log(x^2+y^2) - \log|x|$ gives the implicit solution

$$\arctan\frac{y}{x} - \tfrac{1}{2}\log(x^2+y^2) = C.$$

(c) $(x^2+y^2)\,\mathrm{d}x - 2xy\,\mathrm{d}y = 0$, $y(1) = 0$. This is the worked example $y' = (x^2+y^2)/(2xy)$, whose general solution is $x^2 - y^2 = cx$. Imposing $y(1) = 0$ gives $1 = c$, so

$$x^2 - y^2 = x, \qquad \text{i.e.} \quad y = \pm\sqrt{x^2-x}.$$

Exercise 4.4 Inhomogeneous second-order ODEs

Solution

In each case write $y = \mathrm{CF} + \mathrm{PI}$.
(a) $y'' - 2y' - 8y = 10e^{-x}$. AE $t^2 - 2t - 8 = (t-4)(t+2)$, roots $4, -2$. Trial Ce^{-x}: $C(1+2-8) = -5C = 10 \Rightarrow C = -2$.

$$y = Ae^{4x} + Be^{-2x} - 2e^{-x}.$$

(b) $y'' - 2y' - 8y = 12e^{4x}$. Here 4 is a (simple) root, so trial Cxe^{4x} gives $6Ce^{4x} = 12e^{4x} \Rightarrow C = 2$.

$$y = Ae^{4x} + Be^{-2x} + 2xe^{4x}.$$

(c) $y'' - 4y' + 4y = 6e^{2x}$. AE $(t-2)^2$, double root 2, so trial Cx^2e^{2x} gives $2C = 6 \Rightarrow C = 3$.

$$y = (A+Bx)e^{2x} + 3x^2e^{2x}.$$

(d) $y'' - 6y' + 5y = 6\cos x + 22\sin x$. AE $(t-1)(t-5)$, roots $1, 5$. Trial $a\cos x + b\sin x$ gives $(4a-6b)\cos x + (6a+4b)\sin x$; matching, $4a - 6b = 6$, $6a + 4b = 22 \Rightarrow a = 3$, $b = 1$.

$$y = Ae^{x} + Be^{5x} + 3\cos x + \sin x.$$

(e) $y'' + 9y = 24\cos 3x - 6\sin 3x$. AE roots $\pm 3i$ *resonate* with the forcing, so trial $x(a\cos 3x + b\sin 3x)$ gives $-6a\sin 3x + 6b\cos 3x$; matching, $6b = 24$, $-6a = -6 \Rightarrow a = 1$, $b = 4$.

$$y = A\cos 3x + B\sin 3x + x\cos 3x + 4x\sin 3x.$$

(f) $y'' + 8y' + 16y = 16x^2 + 26$. AE $(t+4)^2$. Trial $ax^2 + bx + c$ gives $16a = 16$, $16b + 16a = 0$, $2a + 8b + 16c = 26 \Rightarrow a = 1$, $b = -1$, $c = 2$.

$$y = (A+Bx)e^{-4x} + x^2 - x + 2.$$

Exercise 4.5 Damped mass–spring $y'' + 4y' + 13y = 0$

Solution

(a) AE $t^2 + 4t + 13 = 0 \Rightarrow t = -2 \pm 3i$. Complex roots $\Rightarrow$ **underdamped**.
(b) General solution $y(t) = e^{-2t}(c_1 \cos 3t + c_2 \sin 3t)$.
(c) $y(0) = 1 \Rightarrow c_1 = 1$. Differentiating, $y'(0) = -2c_1 + 3c_2 = 0 \Rightarrow c_2 = \frac{2}{3}$. Hence

$$y(t) = e^{-2t}\left(\cos 3t + \tfrac{2}{3}\sin 3t\right).$$

(d) The oscillation has period $2\pi/3$. Writing the bracket in amplitude form, the amplitude is $\sqrt{c_1^2 + c_2^2} = \sqrt{1 + \frac{4}{9}} = \frac{\sqrt{13}}{3}$, so the solution is a decaying oscillation contained within the envelope $\pm\dfrac{\sqrt{13}}{3}\,e^{-2t}$.

Solutions to Chapters 5–7 will be added as the course progresses.

End of solutions