MATH4120: Mathematical Modelling and Programming

Exercise sheet 1: Building a model

Chapter 1: the modelling cycle, dimensional homogeneity, first-order ODE models and difference equations

SUBMISSION DEADLINE
Friday 16 October 2026, 16:00

Submit a single scanned PDF via Moodle, please show all of your working.

Question 1 [4 marks]

A small sphere of radius R moving slowly through a fluid experiences a drag force

$$F = 6\pi\mu R v,$$

where v is its speed and μ is the *dynamic viscosity* of the fluid.

(a) Determine the dimensions of μ in terms of M, L and T.

(b) The viscosity of water is quoted as 1.0×10^{-3} Pa s. Verify that Pa s has the dimensions you found in (a). Recall that $1\ \text{Pa} = 1\ \text{N m}^{-2}$.

Question 2 [7 marks]

A water butt collects rainwater from a roof at a constant rate Q (in $\text{m}^3\,\text{day}^{-1}$). It also leaks through a small crack at a rate proportional to the volume of water it currently holds, with constant of proportionality k. Let $V(t)$ be the volume of water in the butt at time t days, and assume the butt is empty at $t = 0$ and never overflows.

(a) Write down a word equation for the rate of change of V, and hence formulate an ODE for $V(t)$.

(b) State the dimensions of Q and k, and verify that your equation is dimensionally homogeneous.

(c) Solve the ODE with $V(0) = 0$.

(d) Find the steady-state volume V_∞, and show that the time taken to reach 95% of it is $t = \ln(20)/k$, independent of Q.

(e) Take $Q = 0.02\ \text{m}^3\,\text{day}^{-1}$ and $k = 0.1\ \text{day}^{-1}$. Evaluate V_∞ and the time to reach 95% of it.

Question 3 [6 marks]

A mortgage of £180,000 is taken out at a fixed interest rate of 0.4% per month. Interest is added at the end of each month, after which the borrower makes a fixed repayment of £900. Let B_n denote the amount owed after n months, with $B_0 = 180{,}000$.

(a) Write down a difference equation for B_n and identify a and b in the standard form $B_{n+1} = aB_n + b$.

(b) Find the equilibrium B^* and determine whether it is stable or unstable, justifying your answer.

(c) Write down the general solution and hence find, to the nearest month, when the mortgage is paid off.

(d) What is the smallest monthly repayment for which the debt ever decreases? Interpret your answer financially.

Question 4 [3 marks]

Look again at the water butt of Question 2. Identify *two* assumptions built into that model which would fail in practice, and for each one describe briefly how you could test whether it matters. Write no more than half a page.